Non-life insurance mathematics (STK4540)

Solutions to the problems of the 1.mandatory assignment

Problem 1 (i) By assumption we know that the inter-arrival times $W_i, i \geq 1$ are *i.i.d.* with common distribution $W_1 \sim Exp(\lambda)$. So by Exercises 1, Problem 1 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{T_n}.$$

The observed inter-arrival times are $W_1 = 1$ (i.e. 09/30/1988 is excluded), $W_2 = 3, W_3 = 4, .., W_{12} = 2$. So $n = 12$ (sample size) and $T_n = W_1 + ... + W_{12} = 38$ (nth arrival time of a fire loss for $n = 12$) we get

$$\widehat{\lambda} = \frac{n}{T_n} = \frac{12}{38} \approx 0.315789.$$

(ii) If the data set was large, the graph of the QQ-plot would indicate that the real inter-arrival time distribution is heavier tailed than the assumed one, since it curves down to the right.

(iii) The theoretical mean excess function is given by $e_F(u) = \frac{1}{\lambda} \approx 3.167$ for all u (see Exercises 2, Problem 1). If the data set was large, the graph which is curving up towards the end wouldn't indicate convergence against that level and rather suggest that the real inter-arrival time distribution is more heavy-tailed than the assumed one.

(iv) Because of Exercises 1, Problem 1 we know that $T_n \sim \Gamma(n, \lambda)$. So

$$P(T_5 > 7) = e^{-\lambda 7} \sum_{k=0}^{5} \frac{(\lambda 7)^k}{k!} \approx 0.92636.$$

Problem 2 (i)

$$\overline{F}(x) = x^{-\alpha} L(x),$$

where

$$L(x) := \left(\frac{x}{x+1} \right)^{\alpha} (1 + \alpha \log(x+1)), x > 0.$$

Since

$$\begin{aligned} & \frac{\left(\frac{\lambda x}{\lambda x+1}\right)^{\alpha}}{\left(\frac{x}{x+1}\right)^{\alpha}} \frac{(1 + \alpha \log(\lambda(x+1)))}{(1 + \alpha \log(x+1))} \\ = \; & \frac{\left(\frac{\lambda}{\lambda+\frac{1}{x}}\right)^{\alpha}}{\left(\frac{1}{1+\frac{1}{x}}\right)^{\alpha}} \frac{(1 + \alpha \log(\lambda) + \alpha \log(x+1))}{(1 + \alpha \log(x))} \underset{x \longrightarrow \infty}{\longrightarrow} 1 \end{aligned}$$

for all $\lambda > 0$, it follows that L is slowly varying. Therefore X_1 is regularly varying with index $\alpha > 0$ (see Def. 3.4.4 in the lecture notes).

Hence, by Prop. 3.4.6 and Remark 3.4.8 X_1 is subexponential, too.

(ii) By differentiating the distribution function F we can compute its probability density and obtain the following log-likelihood function with respect to the claim sizes

$$l(\alpha) = 2n\log(\alpha) + \sum_{i=1}^{n}(-(\alpha+1)\log(x_i) + \log\log(x_i)).$$

By solving the equation

$$\frac{d}{d\alpha}l(\alpha) = \frac{2n}{\alpha} - \sum_{i=1}^{n}\log(x_i) = 0$$

we see that the maximum of the likelihood function is attained at

$$\widehat{\alpha} = \frac{2n}{\sum_{i=1}^{n}\log(x_i)} \stackrel{n=12}{=} 1.62724.$$

Problem 3 We know that

$$\Psi(u) \sim \rho^{-1}\overline{F}^{*}_{X_1,I}(u)$$

for $u \longrightarrow \infty$ (Th. 3.4.13).

We want to approximate

$$F^{*}_{X_1,I}(u) = \frac{1}{E^{*}[X_1]}\int_0^u P^{*}(X_1 > y)dy$$

by means of the empirical distribution function F_n.

So we see that

$$E^{*}[X_1] \approx \frac{1}{n}\sum_{i=1}^{n}X_i \stackrel{n=12}{=} 5.70186.$$

On the other hand,

$$P^{*}(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned}\int_0^u P^{*}(X_1 \quad &> \quad y)dy \approx \int_0^u (1 - \frac{1}{n}\sum_{i=1}^{n}1_{(-\infty,y]}(X_i))dy \\ &= \quad u - \frac{1}{n}\sum_{i=1}^{n}(\max(u, X_i) - X_i).\end{aligned}$$

Using the latter, we find that

$$\Psi(25) = 14.7513\%,\ \Psi(100) = 0\%.$$

Problem 4 See the self-explaining hint.

Problem 5 (i) Conditionally on θ, $S(t), t \geq 0$ has the dynamics of a total claim amount with respect to the Cramér-Lundberg model. So conditionally on θ, the NPC condition has to be

$$E[X_1 |\theta] - cE[W_1 |\theta] < 0.$$

However, $E[X_1 |\theta] = E[X_1] = \frac{1}{\gamma}$, since X_1 is independent of θ and $X_1 \sim Exp(\gamma)$. On the other hand, we have that $E[W_1 |\theta] = \frac{1}{\theta}$. So the conditional NPC condition takes the form

$$\frac{1}{\gamma} - c\frac{1}{\theta} < 0.$$

Further, we know from Exercises 2, Prob 3 that

$$P(\inf_{t\geq 0} U(t) < 0 \,|\theta) = \frac{1}{1+\rho} e^{-\gamma \frac{\rho}{1+\rho} u}$$

with probability 1, where $\rho := c\frac{E[W_1 |\theta]}{E[X_1]} - 1 = c\frac{\gamma}{\theta} - 1 > 0$ (with probability 1).

(ii) It follows from (i) that

$$\begin{aligned}\Psi(u) &= E\left[P(\inf_{t\geq 0} U(t) < 0 \,|\theta)\right] = E\left[\frac{\theta}{c\gamma} \exp(-\gamma \frac{c\frac{\gamma}{\theta} - 1}{c\frac{\gamma}{\theta}} u)\right] \\ &= E\left[\frac{\theta}{c\gamma} \exp(\frac{\theta}{c} u)\right] \exp(-\gamma u).\end{aligned}$$

(iii) If e.g. $\sup_{u>0}(E\left[\frac{\theta}{c\gamma} \exp(\frac{\theta}{c} u)\right] / e^{-ru}) < \infty$ for some $r \in (0, \gamma)$, we obtain from (ii) that $\Psi(u)$ has exponential decay to zero for $u \longrightarrow \infty$.

(iv) For $c := (1+\rho)(\theta/\gamma)$, where $\rho > 0$ is deterministic, we see that the above conditional NPC is satisfied. In this case we can argue as before, but obtain that

$$P(\inf_{t\geq 0} U(t) < 0 \,|\theta) = \frac{1}{1+\rho} e^{-\gamma \frac{\rho}{1+\rho} u}$$

with probability 1. So

$$\Psi(u) = \frac{1}{1+\rho} e^{-\gamma \frac{\rho}{1+\rho} u}.$$

Problem 6 Using characteristic functions it can be shown that $S(t), t \geq 0$ has independent and stationary increments. Using the latter property, we find for $0 < t_1 < t_2 < ... < t_n$ that

$$\begin{aligned} & E\left[\exp(-\lambda_1 S(t_1) - ... - \lambda_n S(t_n))\right] \\ = \; & E\left[\exp(-\lambda_1 S(t_1) - \lambda_2 (S(t_1) + (S(t_2) - S(t_1))) - ... - \lambda_n (S(t_1) + (S(t_2) - S(t_1)) + ... + (S(t_n) - S(t_{n-1}\right. \\ = \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n) S(t_1) - (\lambda_2 + ... + \lambda_n)(S(t_2) - S(t_1)) - ... - \lambda_n (S(t_n) - S(t_{n-1})))\right] \\ = \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n) S(t_1))\right] \cdot E\left[\exp(-(\lambda_2 + ... + \lambda_n)(S(t_2) - S(t_1)))\right] \cdot ... \cdot E\left[\exp(-\lambda_n (S(t_n) - S(t_{n-1}\right. \\ = \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n) S(t_1))\right] \cdot E\left[\exp(-(\lambda_2 + ... + \lambda_n)(S(t_2 - t_1)))\right] \cdot ... \cdot E\left[\exp(-\lambda_n (S(t_n - t_{n-1})))\right]. \end{aligned}$$

On the other hand, using monotone convergence, we get that

$$\begin{aligned}
& E\left[\exp(-\lambda_1 S(t_1))\right] \\
= & E\left[\exp(-\lambda_1 \sum_{i=1}^{N(t_1)} (X_i - x)_+)\right] \\
= & E\left[\exp(-\lambda_1 \sum_{i=1}^{N(t_1)} (X_i - x)_+)(\sum_{k\geq 0} 1_{\{N(t_1)=k\}})\right] \\
= & \sum_{k\geq 0} E\left[\exp(-\lambda_1 \sum_{i=1}^{k}(X_i - x)_+) 1_{\{N(t_1)=k\}}\right] \\
= & \sum_{k\geq 0} E\left[\exp(-\lambda_1 \sum_{i=1}^{k}(X_i - x)_+)\right] P(N(t_1) = k) \\
= & \sum_{k\geq 0}(E\left[\exp(-\lambda_1 (X_1 - x)_+)\right])^k \frac{(\lambda t_1)^k}{k!} e^{-\lambda t_1}.
\end{aligned}$$

By applying the probability density of the exponential distribution, we also find that

$$\begin{aligned}
& E\left[\exp(-\lambda_1 (X_1 - x)_+)\right] \\
= & E\left[1_{\{X_1 \leq x\}} \exp(-\lambda_1 \cdot 0)\right] + E\left[1_{\{X_1 > x\}} \exp(-\lambda_1 (X_1 - x))\right] \\
= & P(X_1 \leq x) + e^{\lambda_1 x} E\left[1_{\{X_1 > x\}} \exp(-\lambda_1 X_1)\right] \\
= & 1 - e^{-\gamma x} + e^{\lambda_1 x} \int_x^\infty \exp(-\lambda_1 y)\gamma \exp(-\gamma y) dy \\
= & 1 - e^{-\gamma x} + \gamma e^{\lambda_1 x} \int_x^\infty \exp(-(\lambda_1 + \gamma) y) dy \\
= & 1 - e^{-\gamma x} + \gamma e^{\lambda_1 x} \left(-\frac{1}{(\lambda_1 + \gamma)} \exp(-(\lambda_1 + \gamma) y)\Big|_{y=x}^{\infty} \right) \\
= & 1 - e^{-\gamma x} + \gamma e^{\lambda_1 x} \frac{1}{(\lambda_1 + \gamma)} \exp(-(\lambda_1 + \gamma) x) \\
= & 1 - e^{-\gamma x} + \frac{\gamma}{(\lambda_1 + \gamma)} e^{-\gamma x} \\
= & 1 + (\frac{\gamma}{(\lambda_1 + \gamma)} - 1) e^{-\gamma x}.
\end{aligned}$$

So

$$\begin{aligned}
& E\left[\exp(-\lambda_1 S(t_1))\right] \\
= & \sum_{k\geq 0}(1 + (\frac{\gamma}{(\lambda_1 + \gamma)} - 1) e^{-\gamma x})^k \frac{(\lambda t_1)^k}{k!} e^{-\lambda t_1} \\
= & \exp\left((1 + (\frac{\gamma}{(\lambda_1 + \gamma)} - 1) e^{-\gamma x}) \lambda t_1 - \lambda t_1 \right) \\
= & \exp\left((\frac{\gamma}{(\lambda_1 + \gamma)} - 1) e^{-\gamma x} \lambda t_1 \right).
\end{aligned}$$

On the other hand, consider the total claim amount process

$$\widetilde{S}(t) = \sum_{i=1}^{N^*(t)} X_i,$$

where $N^*(t), t \geq 0$ is a homogeneous Poisson process with intensity $\lambda^* > 0$, which is independent of the claim sizes $X_i, i \geq 1$. Then we can carry out the same calculations as above for $x = 0$ and obtain that

$$\begin{aligned} & E\left[\exp(-\lambda_1 \widetilde{S}(t_1))\right] \\ = \quad & \exp\left((\frac{\gamma}{(\lambda_1 + \gamma)} - 1)\lambda^* t_1 \right). \end{aligned}$$

Choose now $\lambda^* = \lambda e^{-\gamma x}$. Then we see that

$$E\left[\exp(-\lambda_1 \widetilde{S}(t_1))\right] = E\left[\exp(-\lambda_1 S(t_1))\right]$$

for all $\lambda_1, t_1 > 0$. Since also $\widetilde{S}(t), t \geq 0$ is a process with independent and stationary increments, it follows from (1) and the hint that

$$(\widetilde{S}(t_1), ..., \widetilde{S}(t_n)) \stackrel{\text{law}}{=} (S(t_1), ..., S(t_n))$$

for all $0 < t_1 < t_2 < ... < t_n$. The latter (in connection with the Laplace-Stieltjes transform) also implies that

$$(\widetilde{U}(t_1), ..., \widetilde{U}(t_n)) \stackrel{\text{law}}{=} (U(t_1), ..., U(t_n))$$

for all $0 < t_1 < t_2 < ... < t_n$, where $\widetilde{U}(t) := u + ct - \widetilde{S}(t)$ is another risk process. Hence,

$$\Psi(u) = \widetilde{\Psi}(u),$$

where $\widetilde{\Psi}(u)$ is the ruin probability with respect to $\widetilde{U}(t), t \geq 0$ (which also satisfies the Net Profit condition for $\gamma \geq 1$).

We also know from Problem 3, Exercises 2 that

$$\widetilde{\Psi}(u) = \frac{1}{1+\rho} \exp(-\gamma \frac{\rho}{1+\rho} u).$$

So for $u = 25$, $\gamma = \lambda = 1$, $x = 18$ and $\rho = 0.3$ we get that $\Psi(25) = 0.00240166$.