Non-Life Insurance Mathematics (STK4540)
1. Mandatory assignment

2 Problems out of 6 are supposed to be solved.

Deadline: Thursday, 19. October, 14:30 (electronic submission via Canvas).

Problem 1 Consider the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 09/30/1988 until 11/07/1988:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
10/01/1988	3.283052	10/23/1988	1.597161
10/04/1988	25.953860	10/28/1988	1.774623
10/04/1988	1.064774	10/29/1988	1.721384
10/08/1988	4.081633	11/02/1988	2.333629
10/08/1988	2.288376	11/03/1988	4.494232
10/17/1988	1.100266	11/05/1988	8.873114
10/20/1988	2.838509	11/05/1988	2.676131
10/20/1988	3.194321	11/07/1988	1.147294

The fire losses in the above table are given in millions of Danish Krone. Suppose that the claim numbers $N(t)$ are modelled by a Poisson process with intensity $\lambda > 0$.

(i) Use maximum likelihood estimation (MLE) to estimate the parameter λ.

(ii) Sketch the graph of the QQ-plot of data against the distribution of the inter-arrival times $W_i, i \geq 1$ with respect to the estimated parameter λ.

(iii) Sketch the graph of the mean excess plot with respect to the inter-arrival times.

(iv) Use (i) to calculate

$$P(T_5 > 7),$$

where T_i is the $i-$th arrival time.

What would be possible conclusions in (ii) and (iii), if the data set was large ?

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving on day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^n X_i^{(m)}$ arriving on day m.

Problem 2 Assume that the claim sizes $X_i, i \geq 1$ in Problem 1 are *i.i.d.* with a tail distribution given by

$$\overline{F}(x) = \overline{F}_{X_1}(x) = x^{-\alpha}(1 + \alpha \log(x)), x > 1$$

for $\alpha > 0$.

(i) Show that X_1 is regularly varying.

(ii) Compute the maximum likelihood estimator of the parameter α.

Problem 3 Let $S(t)$ be the total claim amount in Problem 1 described by the Crámer-Lundberg model. Denote by $F^* = F^*_{X_1}$ the real claim size distribution. Assume that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, require that F^* has a probability density and that its integrated tail distribution $F^*_{X_1,I}$ is subexponential.

Give a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 20\%$ and the initial capital $u = 25$ and 100 (in millions of Danish Krone) by approximating F^* by the empirical distribution function with respect to the fire insurance data of Problem 1.

Problem 4 Let us cast a glance at the Danish fire insurance data of Problem 1. Suppose that the total claim amount $S(t)$ of fire losses (in millions of Danish Krone) is described by the Crámer-Lundberg model. Further it is assumed that the law F of the claim sizes X_i is approximated by the empirical distribution function

$$F_n(x) := \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,x]}(X_i)$$

for the sample size n of Problem 1.

Calculate the ML-estimator of the jump intensity of the claim number process and use the estimator in Problem 1 to simulate 40 paths of the total claim amount for the time period of 1 year.

Hint: Note that

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i \geq 1} X_i 1_{[0,t]}(T_i),$$

where $T_i = W_1 + ... + W_i$ are the claim times and W_j the inter-arrival times.

Use existing software packages to simulate the i.i.d sequences of random variables (X_i) and (W_i). Otherwise you can proceed as follows:

Simulation of the uniform law on [0,1]: Generate a sequence of integers $(x_n)_{n \geq 1}$ between 0 and $m-1$ by

$$\begin{cases} x_0 = \text{intitial value} \in \{0, 1, ..., m-1\} \\ x_{n+1} = (a \cdot x_n + b) \bmod m, \end{cases}$$

where

$$\begin{cases} a = 31415821 \\ b = 1 \\ m = 10^8 \end{cases}$$

The numbers $U(\omega_0) = \frac{x_0}{m}, U(\omega_1) = \frac{x_1}{m}, U(\omega_2) = \frac{x_2}{m}, ...$ give the outcomes of a $U \sim U(0,1)$.

Simulation of an exponential distribution:

$$W = -\frac{\log(U)}{\lambda}$$

where U is uniformly distributed on $[0,1]$. So for i.i.d $U_i \sim U(0,1)$ we obtain i.i.d $W_i = \frac{\log(U_i)}{\lambda} \sim Exp(\lambda)$.

Simulation of the claim sizes with law F_n: Consider the quantile function $F_n^{\leftarrow}$ of the empirical distribution function F_n given by

$$F_n^{\leftarrow}(x) := \inf\left\{x \in \mathbb{R} : F_n(x) \geq t\right\}, 0 < t < 1.$$

Visualize the average numbers of claims in $(-\infty, x]$, that is $F_n(x)$ for the sample size n to see how the graph of $F_n^{\leftarrow}(x)$ looks like.

Then for i.i.d $U_i \sim U(0,1)$ the sequence $X_i := F_n^{\leftarrow}(U_i)$ becomes i.i.d with common law F_n (as an approximation of F).

Further note that in virtue of the SLLN

$$T_i \longrightarrow \infty \text{ for } i \longrightarrow \infty$$

with probability 1 holds. So $T_i \geq 1$ year for large i with probability 1.

Problem 5 Consider the risk process $U(t) = u + ct - S(t)$ with total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$, where the *i.i.d.*-sequence $X_i, i \geq 1$ of $Exp(\gamma)$ distributed claim sizes is independent of the *mixed homogeneous Poisson process* N, that is by the process

$$N(t) \stackrel{def}{=} \widetilde{N}(\theta \cdot t), t \geq 0,$$

where $\widetilde{N}(t), t \geq 0$ is a homogeneous Poisson process with intensity $\lambda = 1$ and where the mixing variable θ is positive, not constant with probability 1 and independent of $\widetilde{N}$.

(i) Conditionally on θ, determine the NPC and the probability of ruin for this model, that is

$$P(\inf_{t \geq 0} U(t) < 0 \,|\theta)\,.$$

(ii) Apply the results of (i) to determine the ruin probability

$$\Psi(u) = P(\inf_{t \geq 0} U(t) < 0).$$

(iii) Use part (ii) to give conditions under which $\Psi(u)$ decays exponentially fast to zero as $u \longrightarrow \infty$.

(iv) What changes in the above calculations if you choose the premium $p(t) = (1+\rho)(\theta/\gamma)t$ for some $\rho > 0$? This means that you consider the risk process $U(t) = u + ct - S(t)$ with random premium adjusted to θ.

Problem 6 Consider a reinsurance company with risk process $U(t) = u + ct - S(t)$, where the total claim amount $S(t)$ is given by

$$S(t) = \sum_{i=1}^{N(t)} (X_i - x)_+.$$

The latter amount corresponds to a so-called *excess-of-loss treaty*. Here $(a)_+ \stackrel{\text{def}}{=} \max(a, 0)$ for $a \in \mathbb{R}$. Assume that $N(t), t \geq 0$ is a homogeneous Poisson process with intensity $\lambda > 0$, which is independent of the *i.i.d.* claim size sequence $X_i, i \geq 1$. Suppose that $X_1 \sim Exp(\gamma)$ for $\gamma \geq 1$ and that the premium rate c is defined as

$$c = (1 + \rho)\lambda E\left[(X_1 - x)_+\right]$$

for some safety loading $\rho > 0$.

Derive a formula for the ruin probability $\Psi(u) := P(\inf_{t \geq 0} U(t) < 0)$ and calculate $\Psi(u)$ for $u = 25$, $\gamma = \lambda = 1$, $x = 18$ and $\rho = 0.3$.

Hint: Let $Z = (Z_1, ..., Z_n)$ be a random vector with non-negative random variables $Z_i, i = 1, ..., n$. Define for $\lambda_1, ..., \lambda_n \geq 0$

$$\Psi_Z(\lambda_1, ..., \lambda_n) = E\left[\exp(-\lambda_1 Z_1 - ... - \lambda_n Z_n)\right] \qquad \text{(Laplace-Stieltjes transform)}.$$

Consider another random vector $Y = (Y_1, ..., Y_n)$ with non-negative random variables $Y_i, i = 1, ..., n$. Use the following fact:

Z and Y have the same probability distribution if and only if $\Psi_Z(\lambda_1, ..., \lambda_n) = \Psi_Y(\lambda_1, ..., \lambda_n)$ for all $\lambda_1, ..., \lambda_n \geq 0$.