Non-Life Insurance Mathematics (STK4540) Solutions to the 2.mandatory assignment

Problem 1 By Theorem 6.3.1 in Mikosch we have that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

By Exercises 5, Prob. 2 we know that

1.

$$\begin{aligned} \mu &= E[p(\theta)] = E[P(X_1 > K\,|\theta)] = E[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} \approx 0.127466. \end{aligned}$$

2.

$$\begin{aligned} \lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{2\gamma} \approx 0.030973. \end{aligned}$$

3.

$$\begin{aligned} \varphi &= E[Var[X_1\,|\theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} \approx 0.0802453. \end{aligned}$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.030973}{0.0802453 + 10 \cdot 0.030973} \approx 0.79423 \text{ and } \overline{Y} = 0.2.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y} \approx 18.508\%$$

is the estimated probability for $X_1 > 55000$ NOK given θ.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.00637331.$$

Problem 2 (i) We apply Theorem 6.3.1 in Mikosch and get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Here:

1.

$$\mu = E[\mu(\theta)] = E[\theta] \overset{\text{hint}}{=} \frac{1.75}{1.3} \approx 1.34615.$$

2.

$$\lambda = Var[\mu(\theta)] = Var[\theta] \overset{\text{hint}}{=} \frac{1.75}{(1.3)^2} \approx 1.0355.$$

3.

$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] \overset{\text{hint}}{=} \frac{1.75}{1.3} \approx 1.34615.$$

Thus

$$w \overset{n=10}{=} \frac{10 \cdot 1.0355}{1.34615 + 10 \cdot 1.0355} \approx 0.884956 \text{ and } \overline{X} = 1.2.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 1.21681.$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.119128.$$

(ii) We now assume in (i) that $\theta = \exp(Z)$ for $Z \sim Exp(3)$. Then

1.

$$\begin{aligned} \mu &= E[\mu(\theta)] = E[\exp(Z)] = \int_0^\infty \exp(y) 3\exp(-3y)dy \\ &= 3\int_0^\infty \exp(-2y)dy = 3(-\frac{1}{2}\exp(-2y)\Big|_{y=0}^{\infty}) = 1.5 \end{aligned}$$

$$\begin{aligned} \lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[e^{2Z}] - \mu^2 \\ &= \int_0^\infty \exp(2y)3\exp(-3y)dy - \mu^2 \\ &= 3\int_0^\infty \exp(-y)dy - \mu^2 \\ &= 3(-\exp(-y)|_{y=0}^{\infty}) - \mu^2 \\ &= 3 - \mu^2 = 0.75. \end{aligned}$$

3.

$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] = 1.5.$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.75}{1.5 + 10 \cdot 0.75} \approx 0.833333 \text{ and } \overline{X} = 1.2.$$

Hence

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} = 1.25.$$

Problem 3 (i) It follows from Lemma 5.2.3 (1) in Mikosch that

$$f_\theta(y \,|X = k) = \frac{f_\theta(y) P(X = k \,|\theta = y)}{P(X = k)},$$

where

$$P(X = k) = E[P(X = k \,|\theta)] = E[e^{-\theta}\frac{\theta^k}{k!}] = \int_0^\infty e^{-y}\frac{y^k}{k!} f_\theta(y) dy$$

and

$$P(X = k \,|\theta = y) = e^{-y}\frac{y^k}{k!}.$$

(ii) So by (i) we get that

$$\begin{aligned}
m_k &= E[\theta \,|X = k] = \int_0^\infty y \frac{f_\theta(y) e^{-y}\frac{y^k}{k!}}{P(X = k)} dy \\
&= \frac{(k+1)!}{k! P(X = k)} \int_0^\infty e^{-y}\frac{y^{k+1}}{(k+1)!} f_\theta(y) dy \\
&= \frac{(k+1)}{P(X = k)} E[e^{-\theta}\frac{\theta^{k+1}}{(k+1)!}] \\
&= \frac{(k+1)}{P(X = k)} E[P(X = k+1 \,|\theta)] \\
&= \frac{(k+1)}{P(X = k)} P(X = k+1)
\end{aligned}$$

(iii) Consider w.l.o.g. the case $l = 2$:

$$\begin{aligned}
E[\theta^2 \,|X = k] &\overset{(i)}{=} \int_0^\infty y^2 \frac{f_\theta(y) e^{-y}\frac{y^k}{k!}}{P(X = k)} dy \\
&= \frac{(k+2)!}{k! P(X = k)} \int_0^\infty e^{-y}\frac{y^{k+2}}{(k+2)!} f_\theta(y) dy \\
&= \frac{(k+1)(k+2)}{P(X = k)} E[P(X = k+2 \,|\theta)] \\
&= (k+1)\frac{P(X = k+1)}{P(X = k)}(k+2)\frac{P(X = k+2)}{P(X = k+1)} \\
&= m_k m_{k+1}.
\end{aligned}$$

Problem 4 Suppose that $E[X] < \infty$. Then for $X = X_1$ in Problem 1, we obtain that

$$\begin{aligned} E[X_1] &= \int_0^\infty P(X_1 \geq z)dz = \int_0^\infty E[P(X_1 \geq z\,|\theta)]dz \\ &= \int_0^\lambda E[P(X_1 \geq z\,|\theta)]dz + \int_\lambda^\infty E[P(X_1 \geq z\,|\theta)]dz \\ &= \int_0^\lambda 1dz + + \int_\lambda^\infty E[(\frac{\lambda}{z})^\theta]dz \\ &\overset{n\geq\lambda}{\geq} \int_\lambda^n E[(\frac{\lambda}{z})^\theta]dz \end{aligned} \tag{1}$$

On the other hand,

$$E[(\frac{\lambda}{z})^\theta] = \int_0^\infty \lambda^y z^{-y} f_{\Gamma(\gamma,\beta)}(y)dy.$$

So it follows from (1) by inter-changing integrals (Tonelli's theorem) that

$$\begin{aligned} E[X_1] &\geq \int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y) \int_\lambda^n z^{-y} dz dy \\ &= \int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y) \frac{1}{1-y}(n^{1-y} - \lambda^{1-y})dy \\ &\geq -2\int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y)\lambda^{1-y}dy \\ &\quad +n^{1-\frac{1}{2}} \int_0^{1/2} \lambda^y f_{\Gamma(\gamma,\beta)}(y)\frac{1}{1-y}dy \\ &\underset{n\longrightarrow\infty}{\longrightarrow} \infty, \end{aligned}$$

which leads to a contradiction. So $E[X] = \infty$.

Problem 5 Once more we see from Theorem 6.3.1 in Mikosch that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^n X_j.$$

Here:

1.

$$\begin{aligned} \mu &= E[\mu(\theta)] = E[E[\exp(\theta + \tau Z_1)\,|\theta]] \\ &= E[\exp(\theta)E[\exp(\tau Z_1)\,|\theta]] \\ &\overset{\theta, Z_1 \text{ independent}}{=} E[\exp(\theta)E[\exp(\tau Z_1)]] \\ &\overset{\text{hint in Ex4, Prob.4}}{=} E[\exp(\theta)\exp(\frac{1}{2}\tau^2)] \\ &\overset{\text{hint in Ex4, Prob.4}}{=} \exp(\mu' + \frac{1}{2}(\tau^2 + \sigma^2)) \\ &\approx 1.70106. \end{aligned}$$

2.

$$\begin{aligned}\lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[(\mu(\theta))^2] - \mu^2 \\ &= E[(\exp(\theta)\exp(\frac{1}{2}\tau^2))^2] - \mu^2 \\ &= E[(\exp(2\theta)]\exp(\tau^2) - \mu^2 \\ &= \exp(2\mu' + \tau^2 + 2\sigma^2) - \mu^2 \\ &\approx 1.87714.\end{aligned}$$

3.

$$\begin{aligned}\varphi &= E[Var[X_1\,|\theta]] = E[E[X_1^2\,|\theta]] - E[(E[X_1\,|\theta])^2] \\ &= E[E[\exp(2\theta + 2\tau Z_1)\,|\theta]] - E[\exp(2\theta)\exp(\tau^2)] \\ &= E[\exp(2\theta)E[\exp(2\tau Z_1)]] - E[\exp(2\theta)\exp(\tau^2)] \\ &= E[\exp(2\theta)\exp(2\tau^2)] - E[\exp(2\theta)\exp(\tau^2)] \\ &= \exp(2\mu' + 2\sigma^2 + 2\tau^2) - \exp(2\mu' + 2\sigma^2 + \tau^2) \\ &\approx 3.60216.\end{aligned}$$

So

$$w \approx \frac{8 \cdot 1.87714}{3.60216 + 8 \cdot 1.87714} \approx 0.806536$$

and

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 4.46259 \text{ (in 1000 NOK).}$$

Problem 6 (i) We know from Lemma 5.2.3 that

$$f_\theta(y\,|X = k) = \frac{f_\theta(y)P(X_1 = x_1\,|\theta = y)\ldots P(X_1 = x_n\,|\theta = y)}{P(X = (x_1, \ldots, x_n)')},$$

where

$$f_\theta(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}, 0 < y < 1$$

and

$$P(X_1 = x_j\,|\theta = y) = \binom{k}{x_j} y^{x_j}(1-y)^{k-x_j} \text{ (cond. binomial distr.)},$$

$j = 1, \ldots, n, x_j = 0, \ldots, k$.

We find that

$$\begin{aligned}
P(X &= (x_1, ..., x_n)') = \\
P(X_1 &= x_1, ..., X_n = x_n) = \\
E[P(X_1 &= x_1, ..., X_n = x_n \,|\theta)] \overset{X_j, j\geq 1 \ i.i.d. \text{ given } \theta}{=} \\
E[P(X_1 &= x_1 \,|\theta) \ldots P(X_1 = x_n \,|\theta)] = \\
& E[\prod_{j=1}^{n} \binom{k}{x_j} \theta^{x_j}(1-\theta)^{k-x_j}] \\
&= \int_0^1 \prod_{j=1}^{n} \binom{k}{x_j} y^{x_j}(1-y)^{k-x_j} f_\theta(y)dy \\
&= \int_0^1 \prod_{j=1}^{n} \binom{k}{x_j} y^{x_j}(1-y)^{k-x_j} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}dy \\
&= \{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \prod_{j=1}^{n} \binom{k}{x_j}\} \times \\
& \times \int_0^1 y^{(a+\sum_{j=1}^n x_j)-1}(1-y)^{(b+nk-\sum_{j=1}^n x_j)-1} \frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')} dy \frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')} \\
&= \{\frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \prod_{j=1}^{n} \binom{k}{x_j}\} \int_0^1 y^{a'-1}(1-y)^{b'-1} \frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')} dy \\
&= \frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \prod_{j=1}^{n} \binom{k}{x_j},
\end{aligned}$$

where

$$a' := a + \sum_{j=1}^{n} x_j, b' := b + nk - \sum_{j=1}^{n} x_j > 0.$$

Hence,

$$\begin{aligned}
f_\theta(y \,|X = k) &= \frac{f_\theta(y) P(X_1 = x_1 \,|\theta = y) \ldots P(X_1 = x_n \,|\theta = y)}{P(X = (x_1, ..., x_n)')} \\
&= \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1} \prod_{j=1}^{n} \binom{k}{x_j} y^{x_j}(1-y)^{k-x_j}}{\frac{\Gamma(a')\Gamma(b')}{\Gamma(a'+b')} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \prod_{j=1}^{n} \binom{k}{x_j}} \\
&= \frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')} y^{a'-1}(1-y)^{b'-1},
\end{aligned}$$

which is the density of the Beta distribution for the parameters $a', b' > 0$.

(ii)

$$\widehat{\mu}_B = E[\mu(\theta) \,|X_1, ..., X_n] = (E[\mu(\theta) \,|X_1 = x_1, ..., X_n = x_n])|_{x_1 = X_1, ..., x_n = X_n}.$$

On the other hand,

$$\begin{aligned}
& E[\mu(\theta)\,|X_1 = x_1, ..., X_n = x_n] \\
= & \int_0^1 E[X_1\,|\theta = y]\, f_\theta(y\,|X_1 = x_1, ..., X_n = x_n)\, dy \\
& \overset{\text{expectation of a binomial r.v.}}{=} \\
& \int_0^1 (ky)\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')} y^{a'-1}(1-y)^{b'-1} dy \\
= & k\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}\int_0^1 (y^{(a+1)-1}(1-y)^{b'-1}\frac{\Gamma(a''+b')}{\Gamma(a'')\Gamma(b')} dy \frac{\Gamma(a'')\Gamma(b')}{\Gamma(a''+b')} \\
= & k\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}\frac{\Gamma(a'')\Gamma(b')}{\Gamma(a''+b')} \\
& \overset{\Gamma(z+1)=z\Gamma(z),\Gamma(1)=1}{=} \\
& k\frac{a'}{a'+b'} \\
= & k\frac{a+\sum_{j=1}^n x_j}{a+b+nk}
\end{aligned}$$

where $a'' := a' + 1$.

As for the corresponding risk we know from Theorem 5.2.1 in Mikosch that

$$\rho(\widehat{\mu}_B) = E[Var[\mu(\theta)\,|X_1, ..., X_n]].$$

Since X_1 is binomially distributed given θ, we see that

$$\mu(\theta) = E[X_1\,|\theta] = k\theta.$$

So

$$\rho(\widehat{\mu}_B) = k^2 E[Var[\theta\,|X_1, ..., X_n]].$$

From the hint and the fact that the conditional density of θ is that of a Beta distribution with parameters $a', b' > 0$, we get

$$\begin{aligned}
& Var[\theta\,|X_1 = x_1, ..., X_n = x_n] \\
= & a'b'/[(a'+b'+1)(a'+b')^2] \\
= & (a+\sum_{j=1}^n x_j)(b+nk-\sum_{j=1}^n x_j)/[(a+b+nk+1)(a+b+nk)^2].
\end{aligned}$$

Therefore, the latter implies that

$$\begin{aligned}
\rho(\widehat{\mu}_B) &= \frac{k^2}{[(a+b+nk+1)(a+b+nk)^2]} E[(a+\sum_{j=1}^n X_j)(b+nk-\sum_{j=1}^n X_j)] \\
&= \frac{k^2}{[(a+b+nk+1)(a+b+nk)^2]} \\
&\quad \times (a(b+nk)+(b-a)E[\sum_{j=1}^n X_j] - E[(\sum_{j=1}^n X_j)^2]).
\end{aligned}$$

Further,

$$E[\sum_{j=1}^{n} X_j] = nE[E[X_1 \,|\theta]] = nE[k\theta] \overset{\text{hint}}{=} nk\frac{a}{a+b}$$

and

$$\begin{aligned}
& E[(\sum_{j=1}^{n} X_j)^2] \\
= & \sum_{i,j=1}^{n} E[E[X_i X_j \,|\theta]] \\
= & \sum_{i,j=1, i\neq j}^{n} E[E[X_i \,|\theta]\, E[X_j \,|\theta]] + \sum_{i=1}^{n} E[E[X_i^2 \,|\theta]] \\
= & (n^2-n)E[(E[X_1 \,|\theta])^2] + nE[Var[X_1 \,|\theta] + (E[X_1 \,|\theta])^2] \\
= & n^2 E[(E[X_1 \,|\theta])^2] + nE[Var[X_1 \,|\theta]] \\
= & n^2 E[(k\theta)^2] + nE[k\theta(1-\theta)] \\
= & (n^2k^2 - nk)E[\theta^2] + nkE[\theta] \\
= & (n^2k^2 - nk)(Var[\theta] + (E[\theta])^2) + nkE[\theta] \\
& \overset{\text{hint}}{=} (n^2k^2 - nk)(\{ab/[(a+b+1)(a+b)^2]\} + (\frac{a}{a+b})^2) \\
& +nk\frac{a}{a+b}.
\end{aligned}$$

So

$$\begin{aligned}
& \rho(\widehat{\mu}_B) \\
= & \frac{k^2}{[(a+b+nk+1)(a+b+nk)^2]} \\
& \times(a(b+nk) + (b-a)nk\frac{a}{a+b} \\
& -(n^2k^2 - nk)(\{ab/[(a+b+1)(a+b)^2]\} + (\frac{a}{a+b})^2) \\
& -nk\frac{a}{a+b}).
\end{aligned}$$

Problem 7 (i) Define the process

$$Y_t = 1 + \sum_{i=-\infty}^{t-1} \prod_{j=i+1}^{t} A_j, t \in \mathbb{Z}.$$

We want to show that $Y_t, t \in \mathbb{Z}$ solves the recurrence equation

$$Y_t = A_t Y_{t-1} + 1, t \in \mathbb{Z}, \tag{2}$$

and is strictly stationary.

We see that

$$\begin{aligned} A_t Y_{t-1} + 1 &= A_t(1 + \sum_{i=-\infty}^{t-2} \prod_{j=i+1}^{t-1} A_j) + 1 \\ &= A_t + \sum_{i=-\infty}^{t-2} \prod_{j=i+1}^{t} A_j + 1 \\ &= \sum_{i=t-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{i=-\infty}^{t-2} \prod_{j=i+1}^{t} A_j + 1 \\ &= \sum_{i=-\infty}^{t-1} \prod_{j=i+1}^{t} A_j + 1 = Y_t \end{aligned}$$

for all t.

Next, we have to check that the sum in the definition of Y_t converges with probability 1. For this purpose, let us choose a $c \in (0,1)$ such that $E[\log A_1] < \log c < 0$ (since by our assumption $E[\log A_1] < 0$).

On the other hand, the $i.i.d.$-property of A_j and the SLLN imply that

$$|t-i|^{-1} \sum_{j=i+1}^{t} \log A_j \underset{i \longrightarrow -\infty}{\longrightarrow} E[\log A_1]$$

with probability 1, that is for a sure event Ω^*. Hence, for all $\omega \in \Omega^*$ there exists a $i_0(\omega)$ such that for all $i \leq i_0(\omega)$

$$|t-i|^{-1} \sum_{j=i+1}^{t} \log A_j < \log c.$$

So

$$\prod_{j=i+1}^{t} A_j = \exp(|t-i| \, (|t-i|^{-1} \sum_{j=i+1}^{t} \log A_j)) \leq \exp(|t-i| \log c) = c^{|t-i|}$$

for all $i \leq i_0(\omega)$. Therefore it follows for all $-m \leq i_0(\omega)$ $(\leq t-1)$ that

$$\begin{aligned} \sum_{i=-m}^{t-1} \prod_{j=i+1}^{t} A_j &= \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{i=-m}^{i_0(\omega)} \prod_{j=i+1}^{t} A_j \leq \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{i=-m}^{i_0(\omega)} c^{|t-i|} \\ &\overset{\text{geometric series}}{\leq} \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \sum_{r \geq 0} c^r \\ &= \sum_{i=i_0(\omega)-1}^{t-1} \prod_{j=i+1}^{t} A_j + \frac{1}{1-c} < \infty. \end{aligned}$$

So $\sum_{i=-\infty}^{t-1} \prod_{j=i+1}^{t} A_j$ converges with probability 1.

Next, we aim at verifying strict stationarity.

Claim: For all $n \geq 1$ and $m \leq t$ and $h \in \mathbb{Z}$ we have that

$$((A_s)_{m\leq s\leq t}, ..., (A_s)_{m\leq s\leq t+n-1}) \stackrel{d}{=} ((A_s)_{m+h\leq s\leq t+h}, ..., (A_s)_{m+h\leq s\leq t+h+n-1}). \tag{3}$$

Let us w.l.o.g. show this for $n = 2$ and $h > 0$: In this case the characteristic function of the left hand side of (3) is given by

$$\begin{aligned}
& E[\exp(i(\lambda_1 A_m + ... + \lambda_{t-m+1}A_t + \mu_1 A_1 + ... + \mu_{t-m+2}A_{t+1})] \\
= \ & E[\exp(i(\lambda_1 + \mu_1)A_m)]...E[\exp(i(\lambda_{t-m+1} + \mu_{t-m+1}A_t)]E[\exp(i\mu_{t-m+2}A_{t+1})] \\
= \ & E[\exp(i(\lambda_1 + \mu_1)A_{m+h})]...E[\exp(i(\lambda_{t-m+1} + \mu_{t-m+1})A_{t+h})]E[\exp(i\mu_{t-m+2}A_{t+h+1})] \\
= \ & E[\exp(i(\lambda_1 A_{m+h} + ... + \lambda_{t-m+1}A_{t+h} + \mu_1 A_1 + ... + \mu_{t-m+2}A_{t+h+1})]
\end{aligned}$$

for all λ_i, μ_j because of the $i.i.d.$- property of $A_j, j \in \mathbb{Z}$. So (3) holds.

It follows from (3) that

$$\begin{aligned}
(Y_t^m, Y_{t+1}^m) \quad : \quad & = (\sum_{i=-m}^{t-1} \prod_{j=i+1}^{t} A_j, \sum_{i=-m}^{t} \prod_{j=i+1}^{t+1} A_j) \\
& \stackrel{d}{=} (\sum_{i=-m}^{t-1} \prod_{j=i+1}^{t} A_{j+h}, \sum_{i=-m}^{t} \prod_{j=i+1}^{t+1} A_{j+h}) \\
= \quad & (\sum_{i=-m+h}^{t+h-1} \prod_{j=i+1}^{t+h} A_j, \sum_{i=-m+h}^{t+h} \prod_{j=i+1}^{t+h+1} A_j) \\
& \underset{m\longrightarrow\infty}{\longrightarrow} (Y_{t+h}, Y_{t+h+1})
\end{aligned}$$

with probability 1. On the other hand, we know that

$$(Y_t^m, Y_{t+1}^m) \underset{m\longrightarrow\infty}{\longrightarrow} (Y_t, Y_{t+1})$$

with probability 1. Hence,

$$(Y_t, Y_{t+1}) \stackrel{d}{=} (Y_{t+h}, Y_{t+h+1})$$

or (w.l.o.g.) more generally

$$(Y_t, ..., Y_{t+n-1}) \stackrel{d}{=} (Y_{t+h}, ..., Y_{t+h+n-1}), \tag{4}$$

which gives strict stationarity.

In the next step we want to show uniqueness (in the class of strictly stationary solutions): Assume another strictly stationary solution $Y_t^{'}, t \in \mathbb{Z}$. Then $Y_t - Y_t^{'}$ solves the equation

$$Z_t = A_t Z_{t-1}.$$

Iteration of this equation entails that

$$Y_t - Y_t^{'} = A_t...A_{t-i+1}(Y_{t-i} - Y_{t-i}^{'})$$

for all $i \geq 1$. So

$$\left|Y_t - Y_t^{'}\right| = A_t ... A_{t-i+1} \left|Y_{t-i} - Y_{t-i}^{'}\right|.$$

Using the SLLN, we can show as before that

$$A_t ... A_{t-i+1} \underset{i \longrightarrow \infty}{\longrightarrow} 0$$

with probability 1. On the other hand by using (3) and a similar argument as in the case of the proof of strict stationarity we obtain that

$$A_t ... A_{t-i+1} Y_{t-i} \overset{d}{=} A_{t+i} ... A_{t+1} Y_t.$$

Here again

$$A_{t+i} ... A_{t+1} \underset{i \longrightarrow \infty}{\longrightarrow} 0$$

with probability 1. So for all $\varepsilon > 0$ we have that

$$P(A_t ... A_{t-i+1} Y_{t-i} > \varepsilon) = P(A_{t+i} ... A_{t+1} Y_t > \varepsilon) \underset{i \longrightarrow \infty}{\longrightarrow} 0.$$

So

$$A_t ... A_{t-i+1} Y_{t-i} \underset{i \longrightarrow \infty}{\longrightarrow} 0$$

in probability. By assumption we know that $Y_{t-i}^{'} \overset{d}{=} Y_t^{'}$ for all i. So

$$Y_{t-i}^{'} \underset{i \longrightarrow \infty}{\longrightarrow} Y_t^{'}$$

in distribution. Since $A_{t+i} ... A_{t+1} \underset{i \longrightarrow \infty}{\longrightarrow} 0$ with probability 1 it follows that

$$A_t ... A_{t-i+1} Y_{t-i}^{'} \underset{i \longrightarrow \infty}{\longrightarrow} 0$$

in distribution. Using the latter argument once more, we obtain that

$$A_t ... A_{t-i+1} (Y_{t-i} - Y_{t-i}^{'}) \underset{i \longrightarrow \infty}{\longrightarrow} 0$$

in distribution. Since $|\cdot|$ is continuous, we see that

$$\left|Y_t - Y_t^{'}\right| = A_t ... A_{t-i+1} \left|Y_{t-i} - Y_{t-i}^{'}\right| \underset{i \longrightarrow \infty}{\longrightarrow} 0$$

in distribution. Thus for all $\varepsilon > 0$

$$P(\left|Y_t - Y_t^{'}\right| > \varepsilon) = 1 - P(\left|Y_t - Y_t^{'}\right| \leq \varepsilon) = 0.$$

So

$$Y_t = Y_t^{'}$$

with probability 1, which yields uniqueness.

(ii) Because of strict stationarity we know that

$$Y_t \stackrel{d}{=} Y_0 = 1 + \sum_{i=-\infty}^{-1} \prod_{j=i+1}^{0} A_j.$$

On the other hand, since

$$(A_{-m+1}, ..., A_0) \stackrel{d}{=} (A_1, ..., A_m)$$

we see that

$$\begin{aligned}\sum_{i=-m}^{-1} \prod_{j=i+1}^{0} A_j &\stackrel{d}{=} \sum_{i=-m}^{-1} \prod_{j=i+1}^{0} A_{-j+1} \\ &= \sum_{i=1}^{m} \prod_{j=1}^{i} A_j.\end{aligned}$$

So letting $m \longrightarrow \infty$, we get that

$$Y_t \stackrel{d}{=} Y_0 \stackrel{d}{=} \sum_{i\geq 1} \prod_{j=1}^{i} A_j.$$

Therefore we have for $x > 0$ that

$$\begin{aligned}P(Y_t &> x) = P(Y_0 > x) \geq P(\sup_{n\geq 1} \prod_{j=1}^{n} A_j > x) \\ &= P(\sup_{n\geq 1} \sum_{j=1}^{n} \log A_j > \log x) \\ &= P(\sup_{n\geq 1} S_n > \log x),\end{aligned}$$

where

$$S_n := \sum_{j=1}^{n} Z_j$$

for the *i.i.d.* random variables $Z_j := \log A_j$ such that $E[Z_1] < 0$, which corresponds to a NPC condition in ruin theory.

Then we can employ the proof of Theorem 3.2.3 (Lundberg's inequality) to the partial sum S_n and we get for $x \geq 1$ that

$$m(x) := P(\sup_{n\geq 1} S_n > \log x) \leq \exp(-r \log x) = x^{-r}.$$