Non-Life Insurance Mathematics (STK4540)
2. Mandatory assignment

2 Problems out of 7 are supposed to be solved.

Deadline: Thursday, 9. November, 14:30 (electronic submission via Canvas).

Problem 1 Assume the heterogeneity model for the i-th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$ (Gamma distribution) the claim sizes $X_1, ..., X_n$ in the policy are Pareto distributed with parameters (λ, θ), that is

$$P(X_i > x \,|\theta) = (\lambda/x)^{\theta}, x > \lambda.$$

A reinsurance company only considers the values X_i which exceed a known high treshold K. They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company is interested in finding an estimate of $p(\theta) := P(X_1 > K \,|\theta)$.

Calculate the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1 \,|\theta]$ based on the observations $Y_1, ..., Y_n$, when $X_1 = 22$, $X_5 = 80$, $X_{10} = 71$ and $X_j = 2$ for $j = 2, 3, 4, 6, 7, 8, 9$ (in years), $K = 55$ and $\lambda = 1.5$ (in 1000 NOK). Further, suppose that $\beta = \gamma = 2$.

Problem 2 We consider again the heterogeneity model and require that in the i-th policy of an insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ.

(i) Suppose that θ is Gamma distributed with parameters $\gamma, \beta > 0$.

Calculate the linear Bayes estimator $\widehat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X \,|\theta]$ and its corresponding risk for the claim number data $X_1 = 2$, $X_4 = 1, X_5 = 1$, $X_7 = 7, X_{10} = 1$ and $X_j = 0$ (in 1000 NOK) for $j = 2, 3, 6, 8, 9$ (in years) and the parameters $\gamma = 1.75, \beta = 1.3$.

(ii) Assume now that $\theta = \exp(Z)$, where $Z \sim Exp(\lambda)$ (exponential distribution) for $\lambda = 3$.

Compute $\widehat{\mu}_{LB}$ for the same data as in (i).

Problem 3 Assume the heterogeneity model and consider a policy with one observed claim number X and corresponding heterogeneity parameter θ. Require that X given θ is Poisson distributed with parameter θ, where θ has a continuous density f_θ on $(0, \infty)$. Note that $E[X \,|\theta] = \theta$.

(i) Find the conditional density $f_\theta(y \,|X = k)\,, k \geq 0$ of θ given X and use this to calculate the Bayes estimator $m_k := E[\theta \,|X = k]\,, k \geq 0$.

(ii) Show that

$$m_k = (k+1)\frac{P(X = k+1)}{P(X = k)}, k \geq 0.$$

(iii) Verify that

$$E[\theta^l \,|X = k] = \prod_{j=1}^{l-1} m_{k+j}, k \geq 0, l \geq 1.$$

Problem 4 Calculate $E[X]$ under the assumptions of Problem 1.

Problem 5 Consider the i-th policy in a heterogeneity model. Assume that the claim size variables $X_j, j \geq 1$ given θ are log-normally distributed, that is $X_j = \exp(\theta + \tau Z_j), j \geq 1$. Here θ is normally distributed with mean μ and variance σ^2, $\tau > 0$ is a constant and $Z_j, j \geq 1$ is an $i.i.d.$-sequence of standard normally distributed random variables being independent of θ.

Calculate the linear Bayes estimator with respect to the premium of a car insurance and for the claim size data $X_1 = 15$, $X_6 = 20$, $X_j = 1$ (in 1000 NOK) for $j = 2, 3, 4, 5, 7, 8$ (in years) and the parameters $\mu = 0, \sigma^2 = \frac{1}{2}, \tau = \frac{3}{4}$.

Problem 6 Consider the heterogeneity model with heterogeneity parameter θ and claim numbers $X = X_i = (X_1, ..., X_n)'$ in the ith policy. Assume that θ is Beta distributed with density

$$f_\theta(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}, 0 < y < 1, a, b > 0.$$

Given θ, the claim numbers $X_1, ..., X_n$ are $i.i.d.$ $Bin(k, \theta)$ distributed (Binomial distribution).

(i) Compute the conditional density $f_\theta(y \,|X = x)$ of θ given

$$X = (X_1, ..., X_n)' = (x_1, ..., x_n)'.$$

(ii) Calculate the Bayes estimator $\widehat{\mu}_B$ of $\mu(\theta) = E[X_1 \,|\theta]$ and the corresponding risk.

Hint: $E[\theta] = \frac{a}{a+b}$ and $Var[\theta] = ab/[(a + b + 1)(a + b)^2]$, if θ is Beta distributed with parameters $a, b > 0$.

Problem 7 Consider the stochastic recurrence equation

$$Y_t = A_t Y_{t-1} + B_t, t \in \mathbb{Z}, \tag{1}$$

where $(A_t, B_t) \in \mathbb{R}^2, t \in \mathbb{Z}$ is an $i.i.d.$ sequence.

In the context of insurance Y_t can be interpreted as the present value of future accumulated payments. Here A_t stands for a stochastic discount factor and B_t is a payment at time t. The recurrence relation can be also used in connection with the modelling og financial log-returns (e.g. in ARCH(1)- or GARCH(1,1)-models).

Assume $B_t = 1$ for all t, $E[\log(A_1)] < 0$ and that

$$E[(A_1)^r] = 1$$

for a (unique) positive r.

(i) Show that the unique solution to (1) is given by the process

$$Y_t = 1 + \sum_{i=-\infty}^{t-1} \prod_{j=i+1}^{t} A_j, t \in \mathbb{Z}$$

and *strictly stationary*, that is for all $n \geq 1, t, h \in \mathbb{Z}$:

$$(Y_t, ..., Y_{t+n-1}) \stackrel{d}{=} (Y_{t+h}, ..., Y_{t+h+n-1}).$$

(ii) Prove that the tail distribution of Y_t has a lower bound $m(x) \leq x^{-r}$ for $x \geq 1$.