Non-Life Insurance Mathematics
Exercises 2, 14.09.2023

Problem 1 Consider once again the Danish fire insurance data in the period from 09/01/1983 until 09/25/1983 in Exercises 1, Problem 4.

(i) Compute the mean excess function $e_F(u)$ for $F \sim Exp(\widehat{\lambda})$, where $\widehat{\lambda}$ is the maximum likelihood estimator from Exercises 1, Problem 4.

(ii) Sketch the graph of the ME-plot.

Hint: Let Y be a non-negative random variable with finite mean and distribution $0 < F(x) < 1$ for all $x \geq 0$. The mean excess function of F is defined as

$$e_F(u) = E[Y - u \,|Y > u]\,, u > 0.$$

If $e_F(u)$ converges to infinity for $u \longrightarrow \infty$, it is reasonable to call F heavy-tailed. However, if this limit is finite, we could call F light-tailed.

If $X_i, i \geq 1$ are i.i.d. non-negative random variables with distribution $0 < F(x) < 1$ for all $x \geq 0$, it turns out that for all $u > 0$

$$e_{F_n}(u) \longrightarrow e_F(u)$$

for $n \longrightarrow \infty$ with probability 1, where $e_{F_n}(u)$ is the mean excess function of the empirical distribution function

$$F_n(y) = \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,y]}(X_i),$$

which is also called empirical mean excess function.

One shows that

$$e_{F_n}(u) = \frac{\sum_{i=1}^{n} \max(X_i - u, 0)}{\#\left\{i \leq n : X_i > u\right\}}$$

for all sample sizes $n \geq 1$. # stands for the number of elements in a set.

The mean excess plot (ME-plot) is given by the graph

$$\{(X_{(k)}, e_{F_n}(X_{(k)})) : k = 1, ..., n-1\}.$$

Problem 2 Let us cast a glance at the following Danish fire insurance data collected at Copenhagen Reinsurance in the period between 01/06/1990 and 06/15/1990:

Date	Loss in DKM
06/03/1990	2.475248
06/04/1990	1.815182
06/06/1990	1.495050
06/06/1990	2.042904
06/07/1990	1.070957
06/07/1990	1.237624
06/11/1990	2.062706
06/14/1990	1.017327
06/15/1990	1.650165
06/15/1990	15.284653

The fire losses in the table are based on millions in Danish Krone.

Use the *bootstrap method* to calculate the third moment of the claim sizes, that is

$$E[|X_1^*|^3],$$

where X_1^* has a distribution given by the empirical distribution function F_n with sample size n .

Hint: Set $X_1^* = F_n^{\leftarrow}(U)$, where $U \sim U(0,1)$ (uniform distribution) and $F_n^{\leftarrow}(t)$ is the quantile function, which can be computed as $F_n^{\leftarrow}(t) = X_{(k)}$, if $t \in (\frac{k-1}{n}, \frac{k}{n}]$ and $F_n^{\leftarrow}(t) = X_{(n)}$, if $t \in (\frac{n-1}{n}, 1)$.

Further use the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving at day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^n X_i^{(m)}$ arriving at day m.

Problem 3: Retain the conditions of Problem 1 with the claim size distribution $F \sim Exp(\widehat{\lambda})$.

Derive a formula for the ruin probability $\Psi(u)$ and calculate its value for the safety loading $\rho = 10\%$ and initial capital $u = 4$ (in millions of Danish Krone).

Problem 4 Let F be the Pareto distribution, that is

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\alpha}, \ x > 0$$

for parameters $\alpha, \kappa > 0$.

Show that a Pareto distributed random variable is regularly varying.