Non-Life Insurance Mathematics
Exercises 5, 19.10.2023

Problem 1 Consider the heterogeneity model and assume that in the i-th insurance policy the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ. Suppose that θ is a positive random variable with known moments $m_k := E[\theta^k] < \infty, k = 1, 2, 3, 4$.

(i) Find the linear Bayes estimator $\widehat{\mu}_{LB}$ of $\mu(\theta)$ based on the observation X_1 in terms of X_1, m_1 and m_2. Calculate the corresponding risk.

(ii) Determine the linear Bayes estimator of $\mu(\theta)$ based on the data

$$Y := (X_1, X_1(X_1 - 1))'.$$

Hint: Use the fact that

$$E[Z(Z-1)...(Z-k+1)] = \lambda^k, k \geq 1$$

for Poisson distributed random variables Z with intensity λ.

Problem 2 Consider Problem 3, Exercises 4. Calculate the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1 \,|\theta]$ based on the claim size data $X_1, ..., X_n$, where $Y_1 := 1_{(K,\infty)}(X_1)$ and K is a known high treshold K. Compare the Bayes and linear Bayes estimators and compute those for the realizations $X_1 = 10$, $X_5 = 20$, $X_j = 1$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8$ (in years) and the parameters $\lambda = \beta = \gamma = 1, K = 100$.

Problem 3 Consider a portfolio with n independent policies.

(i) Assume that the claim numbers $X_{i,t}, t \geq 1$ in the ith policy are independent and $Pois(p_{i,t}\theta_i)$ distributed, given θ_i. Assume that $p_{i,t} \neq p_{i,s}$ for some $s \neq t$. Are the conditions of the Bühlmann-Straub model satisfied ?

(ii) Assume that the claim sizes $X_{i,t}, t \geq 1$ in the ith policy are independent and $\Gamma(\gamma_{i,t}, \beta_{i,t})$ distributed, given θ_i. Give conditions on $\gamma_{i,t}, \beta_{i,t}$ under which the Bühlmann-Straub model is applicable. Identify the parameters μ, φ, λ and $p_{i,t}$.

Problem 4 Assume the Bühlmann-Straub model with r policies, where the claim sizes/claim numbers $X_{i,t}, t \geq 1$ in policy i are independent, given θ_i. Let w_i be the weights satisfying $\sum_{i=1}^n w_i = 1$ and $\overline{X}_i = \frac{1}{p_i} \sum_{t=1}^{n_i} p_{i,t} X_{i,t}$ (weighted sample mean), where $p_i := \sum_{t=1}^{n_i} p_{i,t}$.

(i) Show that

$$\widehat{\mu} = \sum_{i=1}^r w_i \overline{X}_i$$

is an unbiased estimator of $\mu = E[\mu(\theta_i)] = E[E[X_{i,t} \,|\theta_i]]$ and calculate the variance of $\widehat{\mu}$.

(ii) Choose the weights w_i in a way such that $Var[\widehat{\mu}]$ is minimized and compute the minimal value $Var[\widehat{\mu}]$.