Prob. 1 : (i) Calculate the maximum likelihood estimator $\hat{\lambda}$ based on the Danish fire insurance data from 02/09/1987 until 02/22/1987, that is maximize the likelihood function

$$L(\lambda) = \prod_{j=1}^{n} f_{W_i}(\lambda; W_j) \quad \text{w.r.t. } \lambda > 0$$

with i.i.d inter-arrival times $W_i \sim Exp(\lambda)$

$$\Rightarrow \quad L(\lambda) = \prod_{j=1}^{n} \lambda \exp(-\lambda W_j) = \lambda^n \exp(\lambda(\underbrace{W_1 + \dots + W_n}_{= T_n})$$

$\Rightarrow$ L attains max. in $\hat{\lambda} = \frac{n}{T_n}$

$\xrightarrow{\text{data}}$ $n = 10$, $T_n = 13$

$$\Rightarrow \quad \hat{\lambda} = \frac{10}{13} \approx 0.7692$$

(ii) distr. of $T_n$ ?

Since $T_n = W_1 + \dots + W_n$ with i.i.d $W_i \sim Exp(\hat{\lambda})$ we get that

$$P(T_n \le x) = 1 - e^{-\hat{\lambda} x} \sum_{k=0}^{n-1} \frac{(\hat{\lambda} x)^k}{k!}, \quad x \ge 0 \qquad (*)$$

Consider e.g. $n = 2$:

$$P(T_2 \le x) = P(W_1 + W_2 \le x) = E[\,1_{\{0 \le w_1 + w_2 \le x\}}(W_1, W_2)\,]$$

$$= E[F(W_1, W_2)],$$

where $F(w_1, w_2) = 1_{\{0 \le w_1 + w_2 \le x\}}(w_1, w_2)$ and where $(W_1, W_2)$ has joint density

$$g(w_1, w_2) = \hat{\lambda} e^{-\hat{\lambda} w_1} \cdot \hat{\lambda} e^{-\hat{\lambda} w_2}$$

$$\Rightarrow \quad P(T_2 \le x) = \int_0^\infty \int_0^\infty F(w_1, w_2)\, g(w_1, w_2)\, dw_1\, dw_2$$

$$= \int_0^x e^{-\hat{\lambda} w_2} \int_0^{x - w_2} \hat{\lambda}^2 e^{-\hat{\lambda} w_1}\, dw_1\, dW_2$$

$$= 1 - e^{-\hat{\lambda} x} \left( \sum_{k=0}^{1} \frac{(\hat{\lambda} x)^k}{k!} \right)$$

$$\overset{(*)}{\Rightarrow} \quad f_{T_n}(x) = e^{-\hat{\lambda} x} \hat{\lambda}^n \frac{x^{n-1}}{\underset{= (n-1)!}{\Gamma(n)}}, \quad x > 0 \quad \Rightarrow$$

$T_n$ Gamma distr. with shape parameter $\alpha = n$ and scale param. $\lambda = \hat{\lambda}$

Ex. #1

② $\cdot \Longrightarrow$ $E[T_n] = \frac{\kappa}{\lambda} = \frac{n}{\lambda} = \frac{13 \cdot n}{10}$ and

$Var[T_n] = \frac{\kappa}{\lambda^2} = \frac{n}{\lambda^2} = \left(\frac{13}{10}\right)^2 \cdot n$

$\overset{n=1}{\Longrightarrow}$ $E[W_1] = 1.3$ expected inter-arrival time

Prob. 2 : We have to show that

$P(B(t) \le x_1, F(t) \le x_2) = P(B(t) \le x_1) \cdot P(F(t) \le x_2)$, $x_1, x_2 \ge 0$

Let $x_1 < t$ then

$\{B(t) \le x_1\} \underset{T_{N(t)} \le t}{\overset{\text{def. of } B(t),}{=}} \{t - x_1 \le T_{N(t)} \le t\}$

$= \{N(t-x_1, t] \ge 1\}$ = at least one jump of S on $(t-x_1, t]$ (+)

$\overset{(+)}{\Longrightarrow}$ $P(B(t) \le x_1) = 1 - P(N(t-x_1, t] = 0) = 1 - e^{-\lambda x_1}$

If $x_1 \ge t$ then

$P(B(t) \le x_1) = P(\underbrace{t - x_1}_{\le 0} \le T_{N(t)} \le t) = 1$

On the other hand

$\{F(t) \le x_2\} \underset{T_{N(t)+1} > t}{\overset{\text{def. of } F(t),}{=}} \{t < T_{N(t)+1} \le t + x_2\}$

$= \{N(t, t+x_2] \ge 1\}$, $x_2 > 0$ (++)

$\overset{(++)}{\Longrightarrow}$ $P(F(t) \le x_2) = 1 - e^{-\lambda x_2}$, $x_2 \ge 0$

W.l.o.g. let $x_1 < t$, $x_2 > 0$

Then

$P(B(t) \le x_1, F(t) \le x_2) \overset{(+),(++)}{=} P(N(t-x_1, t] \ge 1, N(t, t+x_2] \ge 1)$

N has indep. increm. $P(\underbrace{N(t-x_1, t] \ge 1}_{\overset{(+)}{=} P(B(t) \le x_1)}) \cdot P(\underbrace{N(t, t+x_2] \ge 1}_{\overset{(++)}{=} P(F(t) \le x_2)})$

$\longrightarrow$ Prob. 2 is called <u>inspection paradox</u>, since intuition says that

$P(B(t) \le x_1) < 1 - e^{-\lambda x_1}$, $x_1 < t$

and $P(F(t) \le x_2) < 1 - e^{-\lambda x_2}$, $x_2 > 0$

## Exercises 1

**Prob. 4:**

(c) model assumption on the claim size distr.:

$$F \sim \operatorname{Exp}(\hat{\lambda})$$

$\hat{\lambda}$ maximum likelihood estimator based on the claim size data $x_i$:

$x_1 = 7.733848 \longleftarrow x_{(11)}$

$x_2 = 1.437954 \longleftarrow x_{(3)}$

$x_3 = 1.175275 \longleftarrow x_{(1)}$

$x_4 = 1.501669 \longleftarrow x_{(4)}$

$x_5 = 1.506003 \longleftarrow x_{(5)}$

$x_6 = 5.925473 \longleftarrow x_{(9)}$

$x_7 = 15.692792 \longleftarrow x_{(12)}$

$x_8 = 1.436040 \longleftarrow x_{(2)}$

$x_9 = 3.770965 \longleftarrow x_{(8)}$

$x_{10} = 6.716554 \longleftarrow x_{(10)}$

$x_{11} = 1.104229 \longleftarrow x_{(6)}$

$x_{12} = 2.825732 \longleftarrow x_{(7)}$

$\longrightarrow$ Likelihood function:

$$L(\lambda) = \prod_{i=1}^{n} (\lambda \cdot e^{-\lambda x_i}) = \lambda^n e^{\lambda(\sum_{i=1}^{n} x_i)} \longrightarrow \max \text{ in } \lambda$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{12} x_i} \overset{n=12}{\Longrightarrow} \hat{\lambda} = 0.232889$$

QQ-plot against F:

$$\left\{ \left( x_{(k)}, F^{\leftarrow}\left(\tfrac{k}{13}\right) \right), k = 1, \ldots, 12 \right\}$$

Since

$$F(x) = 1 - e^{-\hat{\lambda} x}, \quad x \geq 0$$

has the inverse

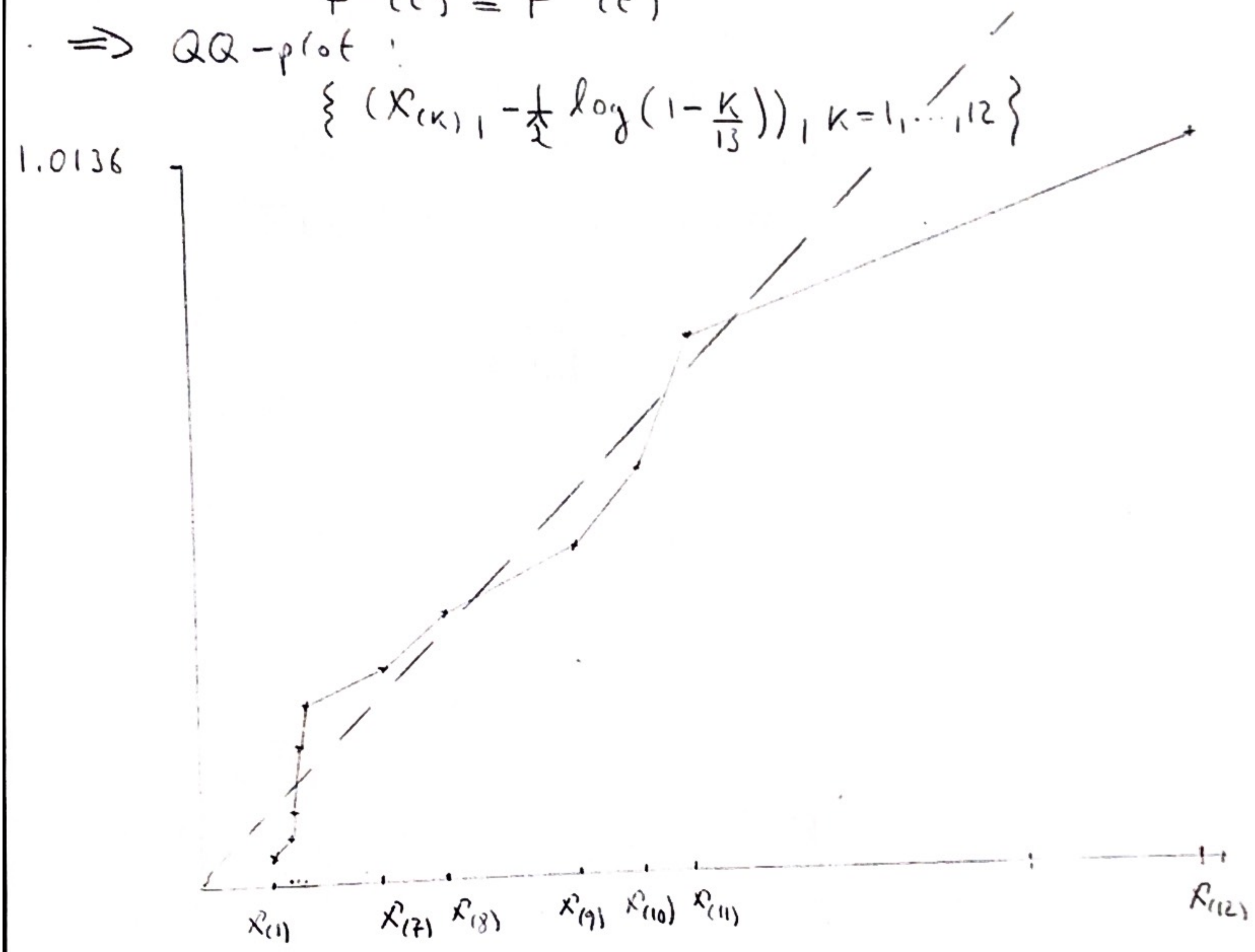
$$F^{-1}(t) = -\frac{1}{\hat{\lambda}} \log(1-t)$$

∴ we have that

$$F^{\leftarrow}(t) = F^{-1}(t)$$

$\Rightarrow$ QQ-plot:

$$\left\{ \left( x_{(k)}, -\frac{1}{\hat{\lambda}} \log\left(1 - \frac{k}{13}\right) \right), k = 1, \dots, 12 \right\}$$




If the data set was large, the graph would indicate a (real) claim size distr., which is heavier tailed than Exp($\hat{\lambda}$)

Exerc. 1

Prob. 5 : The NPC is given by

$$E[X_1] - c\,E[W_1] < 0.$$

Using the empirical mean, we obtain that

$$E[X_1] = 2.2 \text{ and } E[W_1] = 1.3$$

So

$$c > \frac{E[X_1]}{E[W_1]} = 1.692$$