Prob 1 (i) $e_F(u)$? for $F \sim Exp(\hat{\lambda})$

We know that

$$e_F(u) \overset{\text{def}}{=} E[\overset{\sim F}{Y}-u \mid Y>u] \overset{\text{def}}{=} \frac{1}{\underbrace{P(Y>u)}_{=\bar{F}(u)}} E[\overbrace{\mathbb{1}_{(u,\infty)}(Y)\cdot(Y-u)}^{=:Z}]$$

$$= \frac{1}{\bar{F}(u)} \int_0^\infty \underbrace{P(Z>t)}_{=P(Y>u,\, Y>t+u)} dt \overset{\text{subst.}}{=} \frac{1}{\bar{F}(u)} \int_u^\infty \underbrace{P(Y>u,\, Y>t)}_{=\underbrace{P(Y>t)}_{=\bar{F}(t)}} dt$$

$$= \frac{1}{\bar{F}(u)} \int_u^\infty \bar{F}(t)\, dt \qquad (*)$$

$\bar{F}(t) = e^{-\lambda t}$ for $\lambda = \hat{\lambda}$

$$\Rightarrow \int_u^\infty \bar{F}(t)\, dt = \frac{1}{\lambda} e^{-\lambda u}$$

$$\Rightarrow e_F(u) = \frac{1}{\lambda} \approx 4.292 \qquad \hat{\lambda} \approx 0.233$$

(ii) ME-plot based on X_i:

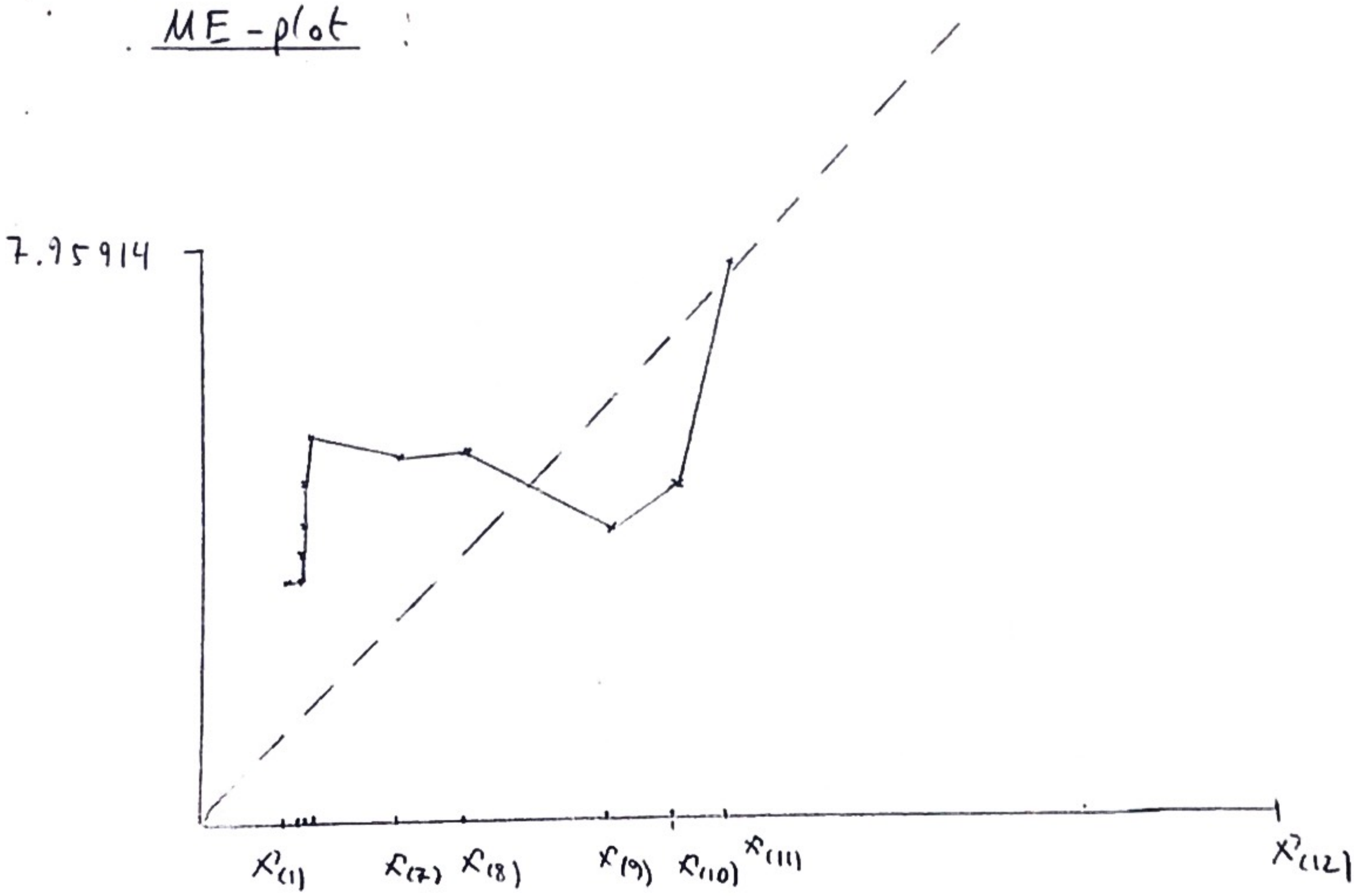
$$\{ (X_{(k)},\, e_{F_n}(X_{(k)})),\ k=1,\dots,n-1 \}$$

for $n=12$

We know from the hint that

$$e_{F_{12}}(X_{(k)}) = \frac{\sum_{i=1}^{12} \max(X_i - X_{(k)}, 0)}{\#\{ i \le 12 : X_i > X_{(k)} \}}$$

ME-plot :



Exercises 2

Prob 2: We want to approximate

$$E[|X_1|^3] \qquad (*)$$

by means of the bootstrap method.
In finding $(*)$ it would be of advantage to know the distribution of

X_1

or equivalently of

$\theta_1(X_1)$

with $\theta_1(x) = x$

$\longrightarrow$ bootstrap method: replace X_1 in the function θ_1 by a r.v. $Y_1 \sim F_n$ with the empirical distr. function

$$F_n(y) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty, y]}(X_i) \quad \leftarrow \text{observed fire losses}$$

for $n = 7$ $\leftarrow$ sample size

By the hint we can e.g. choose

$$Y_1 = F_n^{\leftarrow}(U)$$

with $U \sim U(0,1)$, where $F_n^{\leftarrow}(t)$ is the empirical quantile function

P. 90 $\longrightarrow$ in Mikosch

$$F_n^{\leftarrow}(t) = \begin{cases} X_{(k)}, & t \in \left(\frac{k-1}{n}, \frac{k}{n}\right], \ k = 1, \dots, n-1 \\ X_{(n)}, & t \in \left(\frac{n-1}{n}, 1\right) \end{cases} \qquad (**)$$

$X_{(k)}$ k-th smallest among $X_1, \dots, X_n$

Here: $X_{(1)} = X_6$, $X_{(2)} = X_2$, $X_{(3)} = X_5$, $X_{(4)} = X_4$,
$X_{(5)} = X_1$, $X_{(6)} = X_3$, $X_{(7)} = X_7 = 16.934818$

$\longrightarrow$ 3rd bootstrap moment:

$$E[|Y_1|^3] \qquad (\approx E[|X_1|^3])$$

$$E[|Y_1|^3] = E[|F_7^{\leftarrow}(U)|^3] = \int_0^1 |F_7^{\leftarrow}(t)|^3 \, dt$$

$$= \sum_{k=1}^{6} \int_{\frac{k-1}{7}}^{\frac{k}{7}} \underbrace{|F_7^{\leftarrow}(t)|^3}_{\overset{(**)}{=} |X_{(k)}|^3} dt + \int_{\frac{6}{7}}^{1} \underbrace{|F_7^{\leftarrow}(t)|^3}_{\overset{(**)}{=} |X_{(7)}|^3} dt =$$

$$= \frac{1}{7} \sum_{k=1}^{7} x_{(k)}^3 = \frac{1}{7} \sum_{k=1}^{7} x_k^3 \approx 706.324$$

Exerc. 2

Prob. 3: $F \sim Exp(\hat{\lambda}) \longrightarrow \psi(u)$?

for $\rho = 10\%$ and $u = 4$ (in millions of Danish Krone)

idea: representation

$$\underset{=1-\psi(u)}{\varphi(u)} = \frac{\rho}{1+\rho}\left[1 + \sum_{n\geq 1}(1+\rho)^{-n} P(X_{I,1} + \ldots + X_{I,n} \leq u)\right], \quad (*)$$

where $X_{I,j}$, $j \geq 1$ i.i.d. with common law

$$F_{X_{1,I}}(x) = \overset{=\lambda}{\frac{1}{E[X_1]}} \int_0^x \underbrace{\overline{F}_{X_1}(y)}_{= e^{-\lambda y} \text{ for } \lambda = \hat{\lambda}} dy$$

$$= 1 - e^{\lambda x}$$

$\Longrightarrow \quad X_{I,j} \sim Exp(\lambda)$

Recall: $Y_1, \ldots, Y_n$ i.i.d. with $Y_1 \sim Exp(\lambda)$. Then

$Y_1 + \ldots + Y_n \sim \Gamma(n, \lambda)$, where

$\Gamma(\alpha, \beta)$ Gamma distr. with param. α and β

$\longrightarrow$ density:

$$f(x) = \beta^{\alpha}(\Gamma(\alpha))^{-1} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

$$\Longrightarrow \quad P(X_{I,1} + \ldots + X_{I,n} \leq u) = \int_0^u \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} dx$$

monotone conv.

$$\Longrightarrow \quad \sum_{n\geq 1}(1+\rho)^{-n} P(X_{I,1} + \ldots + X_{I,n} \leq u)$$

$$= \int_0^u \left(\sum_{n\geq 1} \left(\frac{\lambda}{1+\rho}\right)^n \frac{x^{n-1}}{\underbrace{\Gamma(n)}_{=(n-1)!}}\right) e^{-\lambda x} dx = \int_0^u \frac{\lambda}{1+\rho} e^{\frac{\lambda}{1+\rho}x} e^{-\lambda x} dx$$

$$= \frac{\lambda}{1+\rho} \int_0^u e^{-\frac{\lambda\rho}{1+\rho}x} dx = \underbrace{\frac{\lambda}{1+\rho}\frac{(1+\rho)}{\lambda\rho}}_{=\frac{1}{\rho}}\left(1 - e^{-\frac{\lambda\rho}{1+\rho}u}\right)$$

$$\overset{(*)}{\Longrightarrow} \quad \varphi(u) = \overbrace{\frac{\rho}{1+\rho} + \frac{1}{1+\rho}}^{=1} - \frac{1}{1+\rho} e^{-\frac{\lambda\rho}{1+\rho}u}$$

$$\overset{\varphi(u) = 1-\psi(u)}{\Longrightarrow} \quad \psi(u) = \frac{1}{1+\rho} e^{-\frac{\lambda\rho}{1+\rho}u}$$

$$\xrightarrow[u=4]{\hat{\lambda}=0.233,\ \rho=0.1} \quad \psi(u) = 0.835$$

Exerc. 2

Prob. 4 tail distr. of the Pareto distr. is given by

$$\overline{F}(x) = \left(\frac{\varkappa}{\varkappa + x}\right)^{\alpha}, \quad \alpha, \varkappa > 0, \quad x > 0$$

$$= x^{-\alpha} \underbrace{\left(\frac{\varkappa}{\frac{\varkappa}{x} + 1}\right)^{\alpha}}_{=: L(x)}$$

We see that

$$\frac{L(c \cdot x)}{L(x)} = \left(\frac{\frac{\varkappa}{\frac{\varkappa}{c \cdot x} + 1}}{\frac{\varkappa}{\frac{\varkappa}{x} + 1}}\right)^{\alpha} = \left(\frac{\frac{\varkappa}{x} + 1}{\frac{\varkappa}{c \cdot x} + 1}\right)^{\alpha} \xrightarrow[x \to \infty]{} 1 \quad \forall c > 0$$

$\Rightarrow$ L slowly varying $\Rightarrow$ Pareto distr. r.v. is regularly varying with index α