Exercises 3

Prob. 1 (i) Pareto distr: $F_{X_1}(x) = 1 - \gamma^{\alpha} x^{-\alpha}, x \geq \gamma$

$\longrightarrow$ MLE $\hat{\gamma}, \hat{\alpha}$?

prob density $f_{X_1}(x; \alpha, \gamma) = \alpha \gamma^{\alpha} x^{-(\alpha+1)}$

$\longrightarrow$ likelihood function:

$$L(\alpha, \gamma) = \prod_{i=1}^{n} \alpha \gamma^{\alpha} x_i^{-(\alpha+1)} = \alpha^n \gamma^{n\alpha} \prod_{i=1}^{n} x_i^{-(\alpha+1)}$$

$$\Longrightarrow \log L(\alpha, \gamma) = \ell(\alpha, \gamma) = n \log(\alpha) + n\alpha \log(\gamma) - (\alpha+1) \sum_{i=1}^{n} \log(x_i)$$

$\ell(\alpha, \gamma)$ increasing in $\gamma > 0$.

On the other hand, $x > \gamma \Longrightarrow x_i \geq \gamma$ for all i

$$\Longrightarrow \min(x_1, \dots, x_n) \geq \gamma \Longrightarrow \hat{\gamma} = \min(x_1, \dots, x_n)$$

We also see that

$$\frac{\partial \ell(\alpha, \hat{\gamma})}{\partial \alpha} = \frac{n}{\alpha} + n \log(\hat{\gamma}) - \sum_{i=1}^{n} \log(x_i) = 0$$

$\longrightarrow$ max. w.r.t. α attained in

$$\hat{\alpha} = n \Big/ \left(\sum_{i=1}^{n} \log(x_i) - n \log(\hat{\gamma}) \right)$$

$$\overset{n=12}{\approx} \frac{12}{11.118} \approx 1.079$$

(ii) QQ-plot against $F = F_{X_1}$

$$\left\{ \left(X_{(k)}, F^{\leftarrow}\left(\tfrac{k}{13}\right) \right), k = 1, \dots, 12 \right\}$$

Since

$$F(x) = 1 - (\hat{\gamma})^{\hat{\alpha}} x^{-\hat{\alpha}}, \quad x \geq \hat{\gamma}$$

has the inverse

$$F^{-1}(t) = \hat{\gamma} \cdot \frac{1}{(1-t)^{1/\hat{\alpha}}}$$

we have that

$$F^{\leftarrow}(t) = F^{-1}(t)$$

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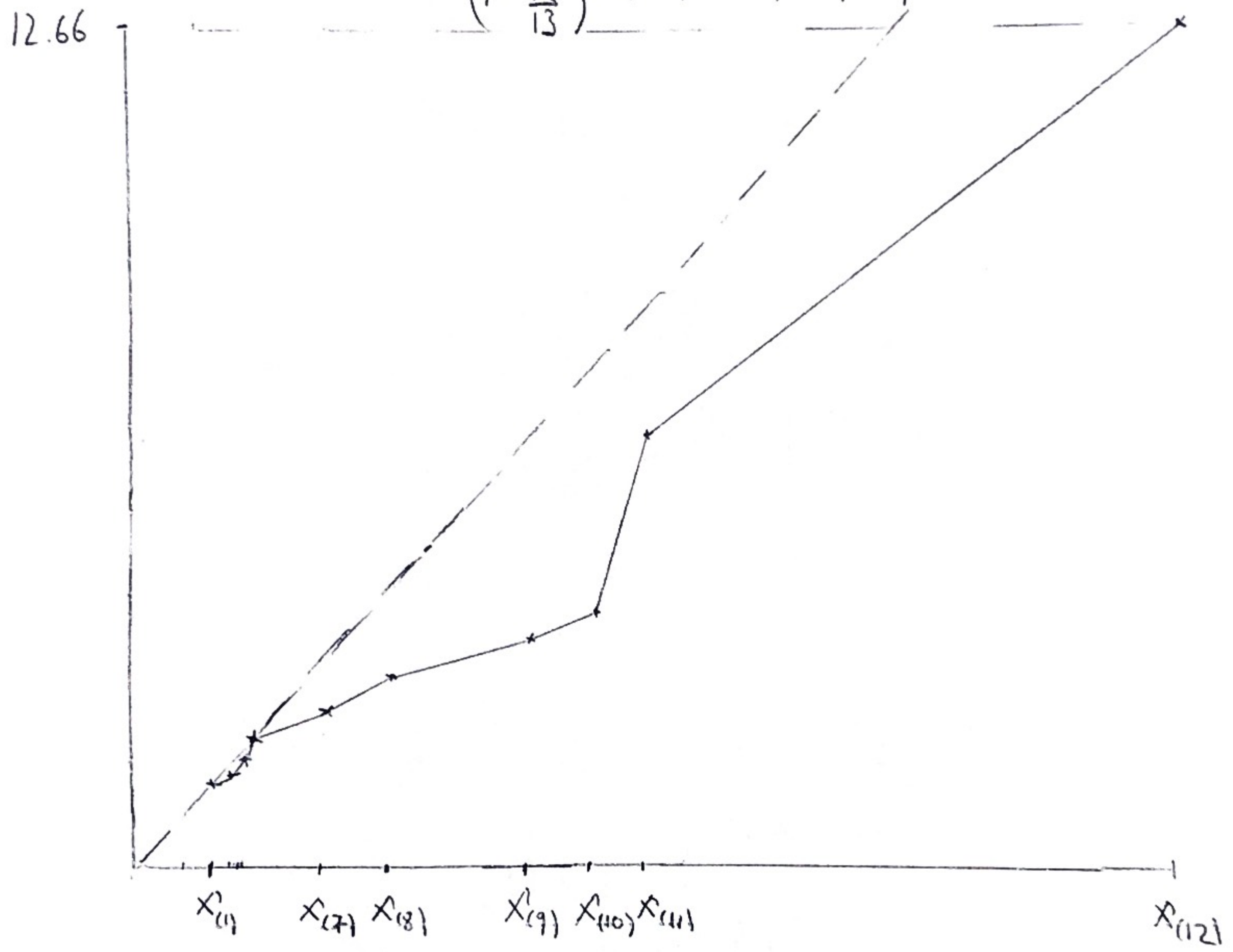
$\Rightarrow$ QQ-plot:

$$\left\{ \left(x_{(k)}, \hat{\gamma} \cdot \frac{1}{\left(1-\frac{k}{13}\right)^{1/\hat{\alpha}}} \right), k = 1, \ldots, 12 \right\}$$



If the data set was large, the graph would indicate that the real claim size distr. is heavier tailed than F_{X_1}

(iii) <u>ME-plot based on x_i</u>:

$$\{ (x_{(k)}, e_{F_n}(x_{(k)})), k = 1, \ldots, n-1 \}$$

See Exerc. 2, Prob. 1

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<u>Prob. 2</u>: We know from Exerc. 2, Prob. 1 that

$$e_F(u) \stackrel{def}{=} E[\overset{\sim F}{Y} - u \mid Y > u]$$

$$= \frac{1}{\bar{F}(u)} \int_u^\infty \bar{F}(x)\, dx \qquad (*)$$

$$F(x) = 1 - (\hat{\delta})^{\hat{\alpha}} x^{-\hat{\alpha}}$$

$$\Rightarrow e_F(u) = (\hat{\delta})^{-\hat{\alpha}} u^{\hat{\alpha}} \int_u^\infty (\hat{\delta})^{\hat{\alpha}} x^{-\hat{\alpha}}\, dx$$

$$= u^{\hat{\alpha}} \left(\frac{1}{-\hat{\alpha}+1} x^{-\hat{\alpha}+1} \Big|_{x=u}^{\infty} \right)$$

$$= \frac{1}{\hat{\alpha}-1} \cdot u = 12.658 \cdot u$$

<u>Prob. 3</u>: We know that

$$\psi(u) \sim \frac{1}{\rho} \bar{F}_{X_1, I}(u) \text{ for large } u$$

$$F_{X_1, I}(u) = \frac{1}{E[X_1]} \int_0^u P(X_1 > x)\, dx$$

$$E[X_1] = \int_{\hat{\delta}}^\infty x \cdot f_{X_1}(x)\, dx$$

$$= \int_{\hat{\delta}}^\infty x \cdot (\hat{\delta})^{\hat{\alpha}} \hat{\alpha}\, x^{-\hat{\alpha}-1}\, dx = (\hat{\delta})^{\hat{\alpha}} \hat{\alpha} \int_{\hat{\delta}}^\infty x^{-\hat{\alpha}}\, dx$$

$$= (\hat{\delta})^{\hat{\alpha}} \hat{\alpha} \underbrace{\left(\frac{1}{-\hat{\alpha}+1} x^{-\hat{\alpha}+1} \Big|_{x=\hat{\delta}}^{\infty} \right)}_{= \frac{1}{1-\hat{\alpha}} \hat{\delta}^{-\hat{\alpha}+1}}$$

$$= \frac{\hat{\alpha}}{\hat{\alpha}-1} \hat{\delta} \approx 16.048$$

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On the other hand,

$$\int_0^u P(X_1>x)\,dx \overset{u\geq\hat{\delta}}{=} \int_0^{\hat{\delta}} \underbrace{P(X_1>x)}_{=1}\,dx + \int_{\hat{\delta}}^u \underbrace{P(X_1>x)}_{=(\hat{\delta})^{\hat{\alpha}}x^{-\hat{\alpha}}}\,dx$$

$$= \hat{\delta} + (\hat{\delta})^{\hat{\alpha}}\left(\frac{1}{-\hat{\alpha}+1}x^{-\hat{\alpha}+1}\Big|_{x=\hat{\delta}}^{u}\right)$$

$$= \hat{\delta} + (\hat{\delta})^{\hat{\alpha}}\frac{1}{\hat{\alpha}-1}\left((\hat{\delta})^{1-\hat{\alpha}} - u^{1-\hat{\alpha}}\right)$$

$$= \hat{\delta} + \frac{\hat{\delta}}{\hat{\alpha}-1} - \frac{(\hat{\delta})^{\hat{\alpha}}}{\hat{\alpha}-1}u^{1-\hat{\alpha}}$$

$\Longrightarrow$ $\psi(4) \approx \varsigma^{-1}\overline{F}_{X_1,I}(4) \approx 8.413 > 1$ (too low u!)

($\psi(4) \approx 83.5\%$ for $X_1 \sim \mathrm{Exp}(\hat{\lambda})$ in Exerc. 2, Prob. 3)

$\psi(12) \approx 7.714 > 1$ (too low u)

($\psi(12) \approx 70.5\%$ for $X_1 \sim \mathrm{Exp}(\hat{\lambda})$ in Exerc. 2, Prob. 3)

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Prob. 4 :

regularly varying with index $\alpha > 1$

$$F_{X_{1,I}}(x) = \frac{1}{E[X_1]} \int_0^x \underbrace{P(X_1 > y)}_{= y^{-\alpha} L(y)} \, dy$$

slowly varying

For subexponentiality it is sufficient to show that $\overline{F}_{X_{1,I}}$ is regularly varying with index $1-\alpha < 0$

So we want to show that

$$\frac{\overline{F}_{X_{1,I}}(c \cdot x)}{\overline{F}_{X_{1,I}}(x)} \xrightarrow[x \to \infty]{} c^{1-\alpha} \quad \text{for all } c > 0$$

$$\frac{\overline{F}_{X_{1,I}}(c \cdot x)}{\overline{F}_{X_{1,I}}(x)} = \frac{1 - \frac{1}{E[X_1]} \int_0^{c \cdot x} \overbrace{f(y)}^{P(X_1 > y)} \, dy}{1 - \frac{1}{E[X_1]} \int_0^x f(y) \, dy}$$

Karamata's $\xrightarrow{}$ Theorem

$$= \frac{\frac{1}{E[X_1]} \int_{c \cdot x}^\infty f(y) \, dy}{\frac{1}{E[X_1]} \int_x^\infty f(y) \, dy} = \frac{\dfrac{\int_{c \cdot x}^\infty f(y) \, dy}{c \cdot x \, f(c \cdot x)}}{\dfrac{\int_x^\infty f(y) \, dy}{x \, f(x)}} \, c \, \frac{f(c \cdot x)}{f(x)} \xrightarrow[x \to \infty]{} c^{1-\alpha}$$

$$\frac{\int_{c \cdot x}^\infty f(y) \, dy}{c \cdot x \, f(c \cdot x)} \xrightarrow[x \to \infty]{} -(1-\alpha)^{-1}, \qquad \frac{\int_x^\infty f(y) \, dy}{x \, f(x)} \xrightarrow[x \to \infty]{} -(1-\alpha)^{-1}, \qquad \frac{f(c \cdot x)}{f(x)} \to c^{-\alpha}$$

$\Longrightarrow$ $X_{0,1} \sim F_{X_{1,I}}$ subexponential.