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Non-Life insurance mathematics

Exerc. 4

Prob. 1: $\hat{\mu}_B = E[\mu(\theta) | X_1, \dots, X_n]$?

Here $\mu(\theta) \overset{def}{=} E[X_1 | \theta]$

For this purpose we want to calculate $E[\mu(\theta) | X_1 = x_1, \dots, X_n = x_n]$

Recall: g function $\Rightarrow$

cond. density of θ given $X_1 = x_1, \dots, X_n = x_n$

$$E[g(\theta) | X_1 = x_1, \dots, X_n = x_n] = \int_0^\infty g(y) \, f_\theta(y | X_1 = x_1, \dots, X_n = x_n) \, dy \quad (*)$$

(which is a consequence of the application of regular conditional prob. in the general case)

We know that

$$P(X_j = k | \theta) = P(X_1 = k | \theta) = e^{-\theta} \frac{\theta^k}{k!}, \; k \geq 0 \quad (**)$$

$\Rightarrow$ X_j Poisson distr. with (random) intensity θ

$\Rightarrow$ $E[X_1 | \theta] = \theta$.

$$\overset{(*)}{\Longrightarrow} E[\underbrace{\mu(\theta)}_{=\theta} | X_1 = x_1, \dots, X_n = x_n] = \int_0^\infty \overbrace{y}^{g(y)} f_\theta(y | X_1 = x_1, \dots, X_n = x_n) \, dy \quad (***)$$

posterior distribution updated by $X_1 = x_1, \dots, X_n = x_n$

w.l.o.g. $n = 2$ (sample size)

$$\overset{L.4.1.5(i)}{\Longrightarrow} f_\theta(y | X_1 = x_1, X_2 = x_2) = \frac{f_\theta(y) \, P(X_1 = x_1 | \theta = y) \, P(X_1 = x_2 | \theta = y)}{P(X_1 = x_1, X_2 = x_2)} \quad (+)$$

prior distribution

1. $f_\theta(y) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma - 1} e^{-\beta x}, \; x > 0$ (Gamma distr.)

2. $P(X_1 = x | \theta = y) \underset{(**)}{\overset{(4.3) \text{ in lect. notes}}{=}} e^{-y} \frac{y^x}{x!}$

3. $P(X_1 = x_1, X_2 = x_2) \overset{E[E[Z|\mathcal{A}]] = E[Z]}{=} \overset{(4.2)}{=} E[E[1_{\{X_1 = x_1, X_2 = x_2\}} | \overbrace{\mathcal{B}(\theta)}^{\mathcal{A}}]] = E[P(X_1 = x_1, X_2 = x_2 | \theta)]$

$P(X_1 = x_1, X_2 = x_2 | \theta) \underset{\text{given } \theta}{\overset{X_j \text{ i.i.d}}{=}} P(X_1 = x_1 | \theta) \, P(X_1 = x_2 | \theta)$

$$\overset{(**)}{=} E\left[e^{-\theta} \frac{\theta^{x_1}}{x_1!} \, e^{-\theta} \frac{\theta^{x_2}}{x_2!}\right]$$

$$= \int_0^\infty e^{-2y} \frac{y^{x_1 + x_2}}{x_1! \, x_2!} \underbrace{\frac{\beta^\gamma}{\Gamma(\gamma)} y^{\gamma - 1} e^{-\beta y}}_{= f_\theta(y)} \, dy$$

Gamma density with $\tilde{\gamma} = \gamma + x_1 + x_2$, $\tilde{\beta} = \beta + 2$

$$= \frac{1}{x_1! \, x_2!} \frac{\beta^\gamma}{\Gamma(\gamma)} \underbrace{\int_0^\infty e^{-(\beta+2) y} \, y^{\gamma + x_1 + x_2 - 1} \cdot \frac{(\beta + 2)^{\gamma + x_1 + x_2}}{\Gamma(\gamma + x_1 + x_2)} \, dy}_{=1} \; \frac{\Gamma(\gamma + x_1 + x_2)}{(\beta + 2)^{\gamma + x_1 + x_2}}$$

② Exerc. 4

$\overset{(+)}{\Longrightarrow}$ $f_\theta(y \mid X_1 = x_1, X_2 = x_2) = \frac{(\beta+2)^{\gamma+x_1+x_2}}{\Gamma(\gamma+x_1+x_2)} e^{-(\beta+2)y} y^{\gamma+x_1+x_2-1}$

$\longrightarrow$ density of a Gamma distr. with $\tilde{\beta} = \beta+2$, $\tilde{\gamma} = \gamma+x_1+x_2$

or for general n:

$f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n)$ density of a Gamma distr

with $\tilde{\beta} = \beta+n$, $\tilde{\gamma} = \gamma + \sum_{j=1}^{n} x_j$

<u>Recall</u>: $Z \sim \Gamma(\gamma, \beta) \Rightarrow E[Z] \left(= \int_0^\infty z f_Z(z)\,dz\right) = \frac{\gamma}{\beta}$

$\overset{(***)}{\Longrightarrow}$ $E[\mu(\theta) \mid X_1 = x_1, \dots, X_n = x_n] = \frac{\tilde{\gamma}}{\tilde{\beta}} = \frac{\gamma + \sum_{j=1}^{n} x_j}{\beta+n}$ (++)

$\overset{(4.3) \text{ in lect. notes}}{\Longrightarrow}$ $\hat{\mu}_B = E[\mu(\theta) \mid X_1, \dots, X_n] = \frac{\gamma + \sum_{j=1}^{n} X_j}{\beta+n}$

$\xrightarrow[\gamma=2,\ \beta=1,\ n=8]{x_1=1,\ x_2=0,\ \dots}$ $E[\mu(\theta) \mid X_1 = x_1, \dots, X_8 = x_8] = \frac{2+4}{1+8} = \frac{2}{3} \approx 0.67$

estimated expected claim number given θ per year