③ **Prob. 2** Exerc. 4

: We want to show w.r.t. the heterogeneity model that

$$E[\mu(\theta_i) \mid X_1, \dots, X_r] = E[\mu(\theta_i) \mid X_i] \qquad (*)$$

Here $\mu(\theta_i) \overset{\text{def}}{=} E[X_{i1} \mid \theta_i]$,

$X_j \overset{\text{def}}{=} (X_{j1}, \dots, X_{jn_j})$, sample size w.r.t. the j-th policy

θ_i indep. of X_j, $j = 1, \dots, r$, $j \neq i$

(as a consequence of (ii) in Def. 4.2.1)

We need the following fact:

Y r.v., $\mathcal{A} := \sigma(Y)$ entirety of information generated by Y. Then Z r.v. on $(\Omega, \mathcal{A}, P)$ $\iff$ there ex. a function g s.t.

$$Z = g(Y) \qquad (1)$$

$\overset{(1)}{\Longrightarrow}$ ($Y = \theta_i$, $Z = \mu(\theta_i)$) $\mu(\theta_i) = g(\theta_i)$ for a function g ($g(y) = E[X_{i1} \mid \theta_i = y]$!)

$\mathcal{F} := \sigma(Y)$ with $Y = (X_1, \dots, X_r)$

Def. 2.10 $\Longrightarrow$ $E[\mu(\theta_i) \mid X_1, \dots, X_r] \overset{\text{def}}{=} E[\mu(\theta_i) \mid \mathcal{F}]$

is the unique r.v. s.t.

$$E\big[E[\mu(\theta_i) \mid X_1, \dots, X_r] \cdot \mathbb{1}_A\big] = E[\mu(\theta_i) \cdot \mathbb{1}_A]$$

for all $A \in \mathcal{F}$

$A \in \mathcal{F} \Longrightarrow Z = \mathbb{1}_A$ r.v. on $(\Omega, \mathcal{F}, P)$

$\overset{(1)}{\Longrightarrow}$ $Z = f(X_1, \dots, X_r)$

$\Longrightarrow E[\mu(\theta_i)\mathbb{1}_A] = E[g(\theta_i) f(X_1, \dots, X_r)]$

W.l.o.g. $i = 1$

$U := (\theta_1, X_1)$ indep. of $V := (X_2, \dots, X_r)$ (because of (ii) in Def. 4.2.1)

$h(u_1, u_2, v_1, \dots, v_{r-1}) := g(u_1) \cdot f(u_1, v_1, \dots, v_{r-1})$

$\Longrightarrow E[\mu(\theta_i)\mathbb{1}_A] = E[h(U, V)]$.

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Prob. 2:

$$\Rightarrow E[E[\mu(\theta_i) \mid X_1, \dots, X_r] \cdot \mathbb{1}_A]$$

$$= E[h(U, V)]$$

$$\overset{\text{dobbel-forventning}}{=} E[\underbrace{(E[h(u, V)])}_{= g(\theta_i)\, E[f(u_2, V)]}|_{u = U}]$$

$$= E[g(\theta_i)\,(E[f(u_2, V)])|_{u_2 = X_1}]$$

$$= E[E[g(\theta_i)\,\underbrace{(E[f(u_2, V)])_{u_2 = X_1}}_{\text{based on } X_1} \mid X_1]]$$

$$= E[\underbrace{E[g(\theta_i) \mid X_1]}_{\overset{(1)}{=} \tilde{g}(X_1),\ \tilde{g} \text{ function}} \cdot (E[f(u_2, V)])|_{u_2 = X_1}]$$

$$\overset{\text{dobbel-forventning}}{=} E[\tilde{g}(X_1)\, f(X_1, \dots, X_r)]$$

$$= E[E[\mu(\theta_i) \mid X_1] \cdot \mathbb{1}_A]$$

$$\overset{\text{uniqueness of } E[Z \mid \mathcal{A}]}{\Longrightarrow} E[\mu(\theta_i) \mid X_1, \dots, X_r] = E[\mu(\theta_i) \mid X_1].$$

Recall: X based on inform. in $\mathcal{A}$ ($\Leftrightarrow$ X r.v. w.r.t. $\mathcal{A}$)

Then $E[X \cdot Y \mid \mathcal{A}] = X \cdot E[Y \mid \mathcal{A}]$

Exerc. 4

Prob. 4: $\hat{\mu}_B = E[\mu(\theta)|X_1,\dots,X_n]$?

Here $\mu(\theta) \overset{def}{=} E[X_1|\theta]$,

$\theta \sim N(\mu, \delta^2)$, $\underset{\tau>0}{\ } \sim N(0,1)$, i.i.d.

$X_j \sim \exp(\theta + \tau Z_j)$ (log-normal distr.)

indep.

$$f_\theta(y|X_1=x_1,\dots,X_n=x_n) \overset{L.4.1.5(ii)}{=} \frac{f_\theta(y)\, f_{X_1}(x_1|\theta=y)\cdots f_{X_1}(x_n|\theta=y)}{f_{X_1,\dots,X_n}(x_1,\dots,x_n)}$$

1. $f_\theta(y) = \frac{1}{\sqrt{2\pi\delta^2}}\, e^{-\frac{1}{2\delta^2}(y-\mu)^2}$ (normal distr.)

2. $f_{X_1}(x|\theta=y)$: $P(X_1 \le x|\theta=y) = P(\theta+\tau Z_1 \le \log x|\theta=y)$

$$\overset{\theta, Z_1 \text{ indep.}}{=} P(y+\tau Z_1 \le \log x) = P(\underset{\sim N(0,1)}{Z_1} \le \frac{\log x - y}{\tau}) = \underset{\text{stand. norm. distr.}}{\Phi}\left(\frac{\log(x)-y}{\tau}\right)$$

$$\overset{\text{differentiation}}{\underset{\text{w.r.t. } x}{\Longrightarrow}} f_{X_1}(x|\theta=y) = \frac{1}{x\tau} f_{Z_1}\left(\frac{\log(x)-y}{\tau}\right)$$

$$= \frac{1}{x\tau}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\tau^2}(\log(x)-y)^2\right)$$

3. $f_{X_1,\dots,X_n}(x_1,\dots,x_n)$:

$$P(X_1 \le x_1,\dots,X_n \le x_n) = E[\underbrace{P(X_1\le x_1,\dots,X_n \le x_n|\theta)}_{\overset{X_j \text{ i.i.d. given } \theta}{=} P(X_1\le x_1|\theta)\cdots P(X_1 \le x_n|\theta)}]$$

$$\overset{\frac{\partial}{\partial x_1\cdots\partial x_n}(\cdot)}{\underset{\text{dom. conv.}}{\Longrightarrow}} f_{X_1,\dots,X_n}(x_1,\dots,x_n) = E[\prod_{i=1}^n f_{X_1}(x_i|\theta)]$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^n f_{X_1}(x_i|\theta=y) \underbrace{\frac{1}{\sqrt{2\pi\delta^2}}\, e^{-\frac{1}{2\delta^2}(y-\mu)^2}}_{=f_\theta(y)} dy$$

$$\overset{2.}{=} \frac{1}{\sqrt{2\pi\delta^2}}\left(\prod_{i=1}^n \frac{1}{x_i\tau\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} \underbrace{\exp\left(-\frac{1}{2\tau^2}\sum_{i=1}^n(\log(x_i)-y)^2\right)\exp\left(-\frac{1}{2\delta^2}(\mu-y)^2\right)}_{=:g(y)} dy \quad (*)$$

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Exerc. 4

Prob. 4: $\frac{1}{\tau^2}\sum_{i=1}^{n}\left((\log x_i)^2 - 2\log(x_i)y + y^2\right) + \frac{\mu^2}{\sigma^2} - \frac{2\mu}{\sigma^2}y + \frac{1}{\sigma^2}y^2$

$$= \frac{1}{\tau^2}\sum_{i=1}^{n}(\log x_i)^2 - \frac{2}{\tau^2}\sum_{i=1}^{n}\log(x_i)\,y + \frac{n}{\tau^2}y^2 + \frac{\mu^2}{\sigma^2} - \frac{2\mu}{\sigma^2}y + \frac{1}{\sigma^2}y^2$$

$$= \overbrace{\frac{1}{\tau^2}\sum_{i=1}^{n}(\log x_i)^2 + \frac{\mu^2}{\sigma^2}}^{=:c}$$

$$+ \underbrace{\left(\frac{n}{\tau^2} + \frac{1}{\sigma^2}\right)}_{=:b} y^2 - \underbrace{\left(\frac{2}{\tau^2}\left(\sum_{i=1}^{n}\log x_i\right) + \frac{2\mu}{\sigma^2}\right)}_{=:a} y$$

$$= c + b\left(y^2 - \frac{a}{b}y\right) = c + b\left(y^2 - 2\left(\frac{a}{2b}\right)y + \left(\frac{a}{2b}\right)^2 - \left(\frac{a}{2b}\right)^2\right)$$

$$= c + b\left(\left(y - \frac{a}{2b}\right)^2 - \left(\frac{a}{2b}\right)^2\right)$$

$$= c - \frac{a^2}{4b} + \frac{1}{b^{-1}}\left(y - \frac{a}{2b}\right)^2$$

$$\overset{(*)}{\Longrightarrow} g(y) = \exp\left(-\frac{1}{2}\left(c - \frac{a^2}{4b}\right)\right)\exp\left(-\frac{1}{2b^{-1}}\left(y - \frac{a}{2b}\right)^2\right) \quad (**)$$

$$\cdot \frac{1}{\sqrt{2\pi b^{-1}}} \cdot \sqrt{2\pi b^{-1}}$$

$\int_{-\infty}^{\infty} f(y)\,dy = 1$ ← prob. density

$$\Longrightarrow (*) = \frac{\sqrt{2\pi b^{-1}}}{\sqrt{2\pi\sigma^2}}\prod_{i=1}^{n}\left(\frac{1}{x_i\tau\sqrt{2\pi}}\right)\exp\left(-\frac{1}{2}\left(c - \frac{a^2}{4b}\right)\right)$$

$$\overset{1.,2.,3.}{\Longrightarrow} f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n) =$$

$$\frac{\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}}\, e^{-\frac{1}{2\sigma^2}(y-\mu)^2}\left(\cancel{\prod_{i=1}^{n}\frac{1}{x_i\tau\sqrt{2\pi}}}\right)\exp\left(-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\log(x_i) - y)^2\right)}{\frac{\sqrt{2\pi b^{-1}}}{\cancel{\sqrt{2\pi\sigma^2}}}\cancel{\prod_{i=1}^{n}\left(\frac{1}{x_i\tau\sqrt{2\pi}}\right)}\exp\left(-\frac{1}{2}\left(c - \frac{a^2}{4b}\right)\right)}$$

$$= \frac{1}{\sqrt{2\pi b^{-1}}}\underbrace{e^{-\frac{1}{2\sigma^2}(y-\mu)^2}\exp\left(-\frac{1}{2\tau^2}\sum_{i=1}^{n}(\log(x_i) - y)^2\right.}_{\overset{(*)}{=} g(y)}\exp\left(\frac{1}{2}\left(c - \frac{a^2}{4b}\right)\right)$$

$$\overset{(**)}{=} \frac{1}{\sqrt{2\pi b^{-1}}}\exp\left(-\frac{1}{2b^{-1}}\left(y - \frac{a}{2b}\right)^2\right)$$

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Prob. 4

$\Longrightarrow f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n)$
density of a r.v. $Z \sim \mathcal{N}(\frac{a}{2b}, b^{-1})$ (+)

$\hat{\mu}_B = E[\mu(\theta) \mid X_1, \dots, X_n]$

$\mu(\theta) \overset{def}{=} E[X_1 \mid \theta] = E[e^{\theta + \tau Z_1} \mid \theta]$

$= e^{\theta} E[e^{\tau Z_1} \mid \theta] = e^{\theta} \underbrace{E[e^{\tau \overset{\sim \mathcal{N}(0,1)}{Z_1}}]}_{e^{\frac{1}{2}\tau^2}}$

indep.

$\Longrightarrow E[\mu(\theta) \mid X_1 = x_1, \dots, X_n = x_n]$

$\overset{(x)\text{ in Prob. 1}}{=} \int_{-\infty}^{\infty} e^{y} e^{\frac{1}{2}\tau^2} f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n)\, dy$

$\overset{(+)}{=} e^{\frac{1}{2}\tau^2} E[e^Z]$ with $Z \sim \mathcal{N}(\frac{a}{2b}, b^{-1})$

$E[e^Z] = \exp(\frac{a}{2b} + \frac{1}{2} b^{-1})$

$\Longrightarrow E[\mu(\theta) \mid X_1 = x_1, \dots, X_n = x_n]$

$= \exp(\frac{1}{2}\tau^2 + \frac{a}{2b} + \frac{1}{2b}) = \exp(\frac{1}{2}\tau^2 + \frac{a+1}{2b})$

$$= \exp\left(\frac{1}{2}\tau^2 + \frac{\frac{2}{\tau^2}\left(\left(\sum_{i=1}^{n} \log(x_i)\right) + \frac{2\mu}{\delta^2}\right) + 1}{2\left(\frac{n}{\tau^2} + \frac{1}{\delta^2}\right)}\right)$$

$\xrightarrow[\mu = 0]{n = 8,\ \delta^2 = \tau = 1}$ $E[\mu(\theta) \mid X_1 = x_1, \dots, X_8 = x_8]$

$= \exp\left(\frac{1}{2} + \frac{2\left(\sum_{i=1}^{8} \log x_i\right) + 1}{18}\right) \approx 3140$ NOK per year

(≈ 3.14 in 1000 NOK) per year