(5)

Excerc. 4

Prob. 3 : (i) $f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n)$?

$$\overset{\text{L.4.1.5 (ii)}}{\Longrightarrow} f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n) = \frac{f_\theta(y)\, f_{X_1}(x_1 \mid \theta = y) \cdots f_{X_1}(x_n \mid \theta = y)}{f_{X_1, \dots, X_n}(x_1, \dots, x_n)} \qquad (*)$$

W.l.o.g $n = 2$

1. $f_\theta(y) = \frac{\beta^\gamma}{\Gamma(\gamma)} y^{\gamma-1} e^{-\beta y}$

2. $f_{X_1}(x \mid \theta = y)$: $P(X_1 \leq x \mid \theta) = 1 - \lambda^\theta x^{-\theta}, \ x > \lambda$

$$\Longrightarrow P(X_1 \leq x \mid \theta = y) = 1 - \lambda^y x^{-y}, \ x > \lambda$$

$$\xrightarrow[\text{w.r.t. } x]{\text{differentiation}} f_{X_1}(x \mid \theta = y) = \lambda^y y x^{-(y+1)}, \ x > \lambda$$

3. $f_{X_1, X_2}(x_1, x_2)$: $P(X_1 \leq x_1, X_2 \leq x_2) = E[\underbrace{P(X_1 \leq x_1, X_2 \leq x_2 \mid \theta)}_{\overset{X_j \text{ i.i.d given } \theta}{=} P(X_1 \leq x_1 \mid \theta)\, P(X_2 \leq x_1 \mid \theta)}]$

$$= E[(1 - \lambda^\theta x_1^{-\theta})(1 - \lambda^\theta x_2^{-\theta})]$$

$$\xrightarrow[\text{w.r.t. } x_1, x_2, \text{ dom. conv.}]{\text{differentiation}} f_{X_1, X_2}(x_1, x_2) = \frac{\partial}{\partial x_1 \partial x_2} P(X_1 \leq x_1, X_2 \leq x_2)$$

$$= E[\theta^2 \lambda^{2\theta} (x_1 \cdot x_2)^{-(\theta+1)}]$$

$$\overset{a^b = e^{b \log a}}{=} \frac{1}{x_1 \cdot x_2} \frac{\beta^\gamma}{\Gamma(\gamma)} \int_0^\infty \underbrace{e^{-y(\overbrace{\beta + \log x_1 + \log x_2 - 2\log\lambda}^{=\tilde{\beta}})} y^{\overbrace{(\gamma+2)}^{=\tilde{\gamma}}-1} \frac{\tilde{\beta}^{\tilde{\gamma}}}{\Gamma(\tilde{\gamma})}}_{\text{Gamma density with param } \tilde{\gamma}, \tilde{\beta}} dy \, \frac{\Gamma(\tilde{\gamma})}{\tilde{\beta}^{\tilde{\gamma}}}$$

$$= \frac{1}{x_1 \cdot x_2} \frac{\beta^\gamma}{\Gamma(\gamma)} \frac{\Gamma(\tilde{\gamma})}{\tilde{\beta}^{\tilde{\gamma}}}$$

$$\xrightarrow[\text{1., 2., 3.}]{(*), n=2} (*) = \frac{\frac{\beta^\gamma}{\Gamma(\gamma)} e^{-\beta y} y^{\gamma-1} \lambda^{2y} y^2 (x_1 \cdot x_2)^{-(y+1)}}{\frac{1}{x_1 \cdot x_2} \frac{\beta^\gamma}{\Gamma(\gamma)} \frac{\Gamma(\tilde{\gamma})}{\tilde{\beta}^{\tilde{\gamma}}}}$$

$$= \frac{\tilde{\beta}^{\tilde{\gamma}}}{\Gamma(\tilde{\gamma})} e^{-y(\beta + \sum_{j=1}^{n} \log(\frac{x_j}{\lambda}))} y^{(\gamma+n)-1} \quad \text{for } n = 2$$

$\overset{\text{general } n}{\Longrightarrow}$ $f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n)$ density of a Gamma distr. with $\tilde{\beta} = \beta + \sum_{i=1}^{n} \log(\frac{x_i}{\lambda})$, $\tilde{\gamma} = \gamma + n$

⑥ Exerc. 4

Prob. 3 : (ii) $\hat{\mu}_B = E[p(\theta) \mid X_1, \dots, X_n]$?

Here $p(\theta) \overset{def}{=} E[Y_1 \mid \theta] = P(X_1 > K \mid \theta) = \lambda^{\theta} K^{-\theta}$,

$\theta \sim \Gamma(\gamma, \beta)$

$Y_1 = \mathbb{1}_{(K,\infty)}(X_1) \sim Bin(1, p(y))$ given $\theta = y$,

since $P(Y_1 = 0 \mid \theta = y) = P(X_1 \le K \mid \theta = y) = 1 - \lambda^y K^{-y}$

and $P(Y_1 = 1 \mid \theta = y) = \underbrace{P(X_1 > K \mid \theta = y)}_{= p(y)} = \lambda^y K^{-y}$

It follows from (***) in Prob. 1 and (i) that

$$E[p(\theta) \mid X_1 = x_1, \dots, X_n = x_n] = \int_0^\infty \underbrace{p(y)}_{=\lambda^y K^{-y}} f_\theta(y \mid X_1 = x_1, \dots, X_n = x_n)\, dy$$

def. as in (i)

$$= \int_0^\infty \lambda^y K^{-y} e^{-y\hat{\beta}} y^{\hat{\gamma}-1} \frac{\hat{\beta}^{\hat{\gamma}}}{\Gamma(\hat{\gamma})} dy$$

$$\overset{a^b = e^{b \cdot \log(a)}}{=} \frac{\hat{\beta}^{\hat{\gamma}}}{\Gamma(\hat{\gamma})} \int_0^\infty \underbrace{e^{-y\overbrace{(\hat{\beta} + \log(\frac{K}{\lambda}))}^{=:\beta^*}} y^{\hat{\gamma}-1} \frac{(\beta^*)^{\hat{\gamma}}}{\Gamma(\hat{\gamma})}}_{\text{Gamma density with param. } \beta^*, \hat{\gamma}} dy \, \frac{\Gamma(\hat{\gamma})}{(\beta^*)^{\hat{\gamma}}}$$

$$= \frac{\hat{\beta}^{\hat{\gamma}}}{\Gamma(\hat{\gamma})} \frac{\Gamma(\hat{\gamma})}{(\beta^*)^{\hat{\gamma}}} = \left(\frac{\hat{\beta}}{\beta^*}\right)^{\hat{\gamma}}$$

$$= \left(\frac{\beta + \sum_{i=1}^n \log(x_i/\lambda)}{\beta + \sum_{i=1}^n \log(x_i/\lambda) + \log(\frac{K}{\lambda})} \right)^{\gamma + n}$$

$$\xrightarrow{x_1 = X_1, \dots, x_n = X_n} \hat{\mu}_B$$