Non-life insurance mathematics
Exercises 5

Prob.1 : (c) $\hat{\mu}_{LB}$ of $\mu(\theta)$ based on X_1 ? ($X_{i,1}$)

$\mu(\theta) \overset{def}{=} E[X_1|\theta]$, $X_1 \sim Pois(\theta)$

$m_k := E[\Theta^k] < \infty$, $k=1,\dots,4$ known

$E[X_1^k] < \infty$, $k=1,\dots,4$

$\underset{\text{Th.4.2.9}}{\Longrightarrow}$ heterogeneity model is a special case of the Bühlmann model and

$$\hat{\mu}_{LB} = (1-w)\mu + w\overline{X_i} \text{ , where}$$

$$w = \frac{n_i \lambda}{\varphi + n_i\lambda} \overset{n_i=1}{=} \frac{\lambda}{\varphi+\lambda}$$

1. $\overline{X_i} = X_1$ for $n_i = 1$, $r=1$

2. $\lambda \overset{def.}{=} Var[\mu(\Theta)] = Var[\underbrace{\mu(\Theta)}_{=E[X_1|\Theta]=\Theta}] = Var[\Theta]$ ($X_1 \sim Pois(\theta)$)
$= E[\Theta^2] - (E[\Theta])^2 \overset{def.}{=} m_2 - m_1^2$

3. $\mu \overset{def.}{=} E[X_1] = m_1$

4. $\varphi \overset{def}{=} E[Var[X_1|\theta]] \overset{L.4.2.7(ii)}{=}$
$E[X_1^2] - \lambda - \mu^2 = E[E[X_1^2|\theta]] - \lambda - \mu^2$

Hint: $E[X_1(X_1-1)|\theta] = \theta^2$

$\Longrightarrow E[X_1^2|\theta] = \theta^2 + \underbrace{E[X_1|\theta]}_{=\Theta}$

$\Longrightarrow \varphi = E[E[X_1^2|\theta]] - \lambda - \mu^2 = m_2 + m_1 - \lambda - \mu^2$
$\overset{2.,3.}{=} m_2 + m_1 - m_2 + m_1^2 - m_1^2 = m_1 > 0$

risk : $\varrho(\hat{\mu}_{LB}) = (1-w)\lambda$

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(ii) $\hat{\mu}_{LB}$ of $\mu(\theta)$ based on $X_1, X_1(X_1-1)$?

<u>idea</u>: normal equations of Prop. 4.2.2 applied to

$Y := (X_1, X_1(X_1-1))'$ ← transpose observation

$X := E[X_1|\theta] \overset{def}{=} \mu(\theta)$

$a := (a_1, a_2)$

⟶ $\hat{Y}$ minimizer w.r.t. $\mathcal{L}' \iff \hat{Y} = a_0 + a' \cdot Y$
with (a_0, a) satisfying the normal equations!

$$a_0 = E[X] - a'E[Y] = \underbrace{E[X_1]}_{=m_1} - a_1 \underbrace{E[X_1]}_{=m_1} - a_2 \underbrace{E[X_1(X_1-1)]}_{\substack{=E[E[X_1(X_1-1)|\theta]] \\ \overset{Hint}{=} E[\theta^2] = m_2}}$$

$$= m_1(1-a_1) - a_2 m_2 \qquad (*)$$

and

$$\Sigma'_{X,Y} = a' \cdot \Sigma_Y \qquad (**)$$

$$\Sigma_Y \overset{def}{=} \begin{pmatrix} Var[Y_1] & Cov[Y_1, Y_2] \\ Cov[Y_1, Y_2] & Var[Y_2] \end{pmatrix} =: \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

1. $Var[Y_1] = Var[X_1] = E[X_1^2] - (\underbrace{E[X_1]}_{=m_1})^2$
$= E[\underbrace{E[X_1^2|\theta]}_{\overset{Hint}{=} \theta^2+\theta}] - m_1^2 = m_2 + m_1 - m_1^2$

2. $Var[Y_2] = E[Y_2^2] - \overbrace{(\underbrace{E[Y_2]}_{=E[E[Y_2|\theta]] = m_2})^2}^{=m_2^2}$, with $E[Y_2|\theta] \overset{Hint}{=} \theta^2$

$E[Y_2^2] = E[X_1^2(X_1-1)^2] = E[X_1^4] - 2E[X_1^3] + \underbrace{E[X_1^2]}_{=m_2+m_1}$

$\overset{Hint}{\underset{\text{conditioning w.r.t. } \theta}{=}} (m_4 + 6m_3 + 7m_2 + m_1) - 2(m_3 + 3m_2 + m_1) + m_2 + m_1$

$= m_4 + 4m_3 + 2m_2$

$\Longrightarrow Var[Y_2] = m_4 + 4m_3 + 2m_2 - m_2^2$

3. $Cov[Y_1, Y_2] \overset{def}{=} E[(X_1 - E[X_1])(X_1(X_1-1) - E[X_1(X_1-1)])]$
$= E[X_1^3] - E[X_1^2] - E[X_1] \cdot E[X_1(X_1-1)]$

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③ $\overset{\text{Hint,}}{\underset{\text{conditioning w.r.t. }\theta}{=}} (m_3 + 3m_2 + m_1) - (m_2 + m_1) - m_1 m_2 = m_3 + 2m_2 - m_1 \cdot m_2$

$$\Sigma'_{X,Y} \overset{\text{def.}}{=} ((\mathrm{Cov}[X, Y_1], \mathrm{Cov}[X, Y_2]) \overset{\text{Hint,}}{\underset{\text{conditioning w.r.t. }\theta}{=}}$$

$$= (m_2 - m_1^2, m_3 - m_1 \cdot m_2) =: (\alpha, \beta)$$

$\Longrightarrow$ $(**) \Longleftrightarrow$

and $\alpha = a_1 \cdot a + a_2 \cdot b$

$\beta = a_1 \cdot b + a_2 \cdot c$

$\Longrightarrow$ $a_1 = \dfrac{\alpha - a_2 \cdot b}{a}$, if $a = \mathrm{Var}[Y_1] > 0$

$\Longrightarrow$ $\beta = \left(\dfrac{\alpha - a_2 \cdot b}{a}\right) \cdot b + a_2 \cdot c$

$= \dfrac{\alpha \cdot b}{a} + a_2 \left(c - \dfrac{b^2}{a}\right)$

$\Longrightarrow$ $a_2 = \left(\beta - \dfrac{\alpha \cdot b}{a}\right) / \left(c - \dfrac{b^2}{a}\right)$, if $c - \dfrac{b^2}{a} \neq 0$

$\longrightarrow$ a_0, a_1, a_2 given in terms of $m_1, \ldots, m_4$, if $a, c - \dfrac{b^2}{a} \neq 0$

(1) Exerc. 5

Prob. 3 (i) By assumption we have that $X_{i,t} \sim Pois(P_{i,t}\Theta_i)$, given Θ_i

$$\longrightarrow E[X_{i,t}|\Theta_i] = Var[X_{i,t}|\Theta_i] = P_{i,t}\cdot\Theta_i$$
$$= \frac{(P_{i,t})^2\cdot\Theta_i}{P_{i,t}} \qquad (*)$$

However, by the Bühlmann-Straub model ((B-S-model) there is a function v s.t.

$$Var[X_{i,t}|\Theta_i] = \frac{v(\Theta_i)}{P_{i,t}} \qquad (**)$$

for all i,t

There ex. s,t s.t. $P_{i,t} \neq P_{i,s}$ $\overset{(*),(**)}{\Longrightarrow}$ there exists no such function v $\Longrightarrow$ conditions of this model are not satisfied.

(ii) Recall: $X \sim \Gamma(\gamma,\beta) \Longrightarrow E[X] = \frac{\gamma}{\beta}$, $Var[X] = \frac{\gamma}{\beta^2}$

$$\Longrightarrow E[X_{i,t}|\Theta_i] = \frac{\gamma_{i,t}}{\beta_{i,t}}, Var[X_{i,t}|\Theta_i] = \frac{\gamma_{i,t}}{\beta_{i,t}^2}$$

$\overset{\text{B-S-model}}{\Longrightarrow}$ there ex. functions μ, v s.t.

$$\frac{\gamma_{i,t}}{\beta_{i,t}} = \mu(\Theta_i) \text{ and } P_{i,t}\frac{\gamma_{i,t}}{\beta_{i,t}^2} = v(\Theta_i) \text{ with prob. 1}$$

for all i,t

$$\Longrightarrow \frac{\gamma_{i,1}}{\beta_{i,1}} = \mu(\Theta_i) \text{ and } P_{i,1}\frac{\gamma_{i,1}}{\beta_{i,1}^2} = v(\Theta_i) \text{ with prob. 1}$$

for all i $\qquad (+)$

$$\overset{(+)}{\longrightarrow} \mu \overset{def}{=} E[\mu(\Theta_i)] \underset{i.i.d.}{\overset{\Theta_i, i\geq 1}{=}} E[\mu(\Theta_1)] = E\left[\frac{\gamma_{1,1}}{\beta_{1,1}}\right],$$
$$\lambda \overset{def}{=} Var[\mu(\Theta_i)] = Var[\mu(\Theta_1)] = Var\left[\frac{\gamma_{1,1}}{\beta_{1,1}}\right],$$
$$\varphi \overset{def}{=} E[v(\Theta_i)] = E[v(\Theta_1)] = P_{1,1}\cdot E[(\gamma_{1,1})/(\beta_{1,1})^2]$$

② Exerc. 5

Prob. 4: (i) We want to show that $E[\hat{\mu}] = \mu$

$$E[\hat{\mu}] = \sum_{i=1}^{r} w_i E[\overline{X}_i] = \sum_{i=1}^{r} w_i \frac{1}{P_i} \sum_{t=1}^{n_i} P_{i,t} E[X_{i,t}]$$

$$= \sum_{i=1}^{r} w_i \frac{1}{P_i} \sum_{t=1}^{n_i} P_{i,t} \underbrace{E[E[X_{i,t}|\Theta_i]]}_{=\mu} = \mu \sum_{i=1}^{r} w_i \overbrace{\frac{1}{P_i} \sum_{t=1}^{n_i} P_{i,t}}^{=1}$$

$$= \mu \overbrace{\sum_{i=1}^{r} w_i}^{=1} = \mu .$$

(ii) $$\mathrm{Var}[\hat{\mu}] \overset{\text{def.}}{=} \mathrm{Var}\Big[\sum_{i=1}^{r} w_i \overline{X}_i\Big] \underset{\text{indep.}}{\overset{\overline{X}_i,\, i\geq 1}{=}}$$

$$= \sum_{i=1}^{r} \mathrm{Var}[w_i \overline{X}_i] = \sum_{i=1}^{r} w_i^2 \mathrm{Var}[\overline{X}_i]$$

Recall: law of total variance:

$$\mathrm{Var}[X] = E[\mathrm{Var}[X|\theta]] + \mathrm{Var}[E[X|\theta]]$$

$$\xrightarrow[\theta = \Theta_i]{X = \overline{X}_i,} \mathrm{Var}[\overline{X}_i] = E[\mathrm{Var}[\overline{X}_i|\Theta_i]] + \mathrm{Var}[E[\overline{X}_i|\Theta_i]]$$

1. $$E[\overline{X}_i|\Theta_i] = \frac{1}{P_i} \sum_{t=1}^{n_i} P_{i,t} \overbrace{E[X_{i,t}|\Theta_i]}^{=\mu(\Theta_i) \leftarrow \text{ i.i.d.}}$$

$$\Rightarrow \mathrm{Var}[E[\overline{X}_i|\Theta_i]] = \frac{1}{P_i^2} \sum_{t=1}^{n_i} (P_{i,t})^2 \overbrace{\mathrm{Var}[\mu(\Theta_i)]}^{= \mathrm{Var}[\mu(\Theta_i)] \overset{\text{def}}{=} \lambda}$$

2. $$E[\mathrm{Var}[\overline{X}_i|\Theta_i]] \underset{\text{given } \Theta_i}{\overset{X_{i,t},\, t\geq 0 \text{ indep.}}{=}}$$

$$\frac{1}{P_i^2} E\Big[\sum_{t=1}^{n_i} P_{i,t}^2 \underbrace{\mathrm{Var}[X_{i,t}|\Theta_i]}_{\overset{\text{def}}{=} V(\Theta_i)/P_{i,t}}\Big] = \frac{1}{P_i^2} \sum_{t=1}^{n_i} P_{i,t}^2 \frac{\overset{\text{def.}}{\overbrace{E[V(\Theta_i)]}^{= \varphi}}}{P_{i,t}}$$

$$= \varphi / P_i$$

$$\overset{1., 2.}{\Longrightarrow} \mathrm{Var}[\overline{X}_i] = \frac{\varphi}{P_i} + \lambda \frac{\sum_{t=1}^{n_i} P_{i,t}^2}{P_i^2}$$

$$\Rightarrow \mathrm{Var}[\hat{\mu}] = \sum_{i=1}^{r} w_i^2 \left(\frac{\varphi}{P_i} + \lambda \frac{\sum_{t=1}^{n_i} P_{i,t}^2}{P_i^2} \right)$$

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Prob. 4: $f(w_1,\dots,w_r) := \sum_{i=1}^{r} w_i^2 \underbrace{Var[\bar{X}_i]}_{=: a_i} \overset{\text{def}}{=} Var[\hat{\mu}]$

$g(w_1,\dots,w_r) := \sum_{i=1}^{r} w_i - 1$

We want to find a minimizer of f under the condition $g(w_1,\dots,w_r)=0$, $w_i > 0$, $i=1,\dots,r$

$\longrightarrow \quad \frac{\partial}{\partial w_i} f(w_1,\dots,w_r) + \lambda \frac{\partial}{\partial w_i} g(w_1,\dots,w_r) = 0, \quad i=1,\dots,r$

Lagrange multiplier

$\Longleftrightarrow \quad 2 w_i a_i + \lambda = 0, \quad i=1,\dots,r$

$\Longrightarrow \quad w_i = -\lambda/2a_i \overset{!}{>} 0 \qquad (*)$

$\overset{\sum_{i=1}^r w_i = 1}{\Longrightarrow} \quad 1 = -\frac{\lambda}{2} \sum_{i=1}^{r} \frac{1}{a_i} \Longrightarrow \lambda = \frac{-2}{\sum_{i=1}^{r} \frac{1}{a_i}}$

$\overset{(*)}{\Longrightarrow} \quad w_i = \frac{1}{\sum_{j=1}^{r} \frac{a_i}{a_j}}, \quad i=1,\dots,r \qquad (**)$

$\longrightarrow Var[\hat{\mu}] = f(w_1,\dots,w_r) \overset{(*)}{=} \sum_{i=1}^{r} \frac{\lambda^2}{4a_i^2} a_i$

$= \frac{\lambda^2}{4} \underbrace{\sum_{i=1}^{r} \frac{1}{a_i}}_{= \frac{2}{-\lambda}} = -\frac{\lambda}{2} =$

$= \frac{1}{\sum_{i=1}^{r} \frac{1}{a_i}} < \frac{1}{\frac{1}{a_i}} = a_i = f(0,\dots,0,\overset{=1}{w_i},0,\dots,0)$

for all i

$\Longrightarrow \quad w_i, i=1,\dots,r$ as in $(**)$ is the minimum