(4) Exerc. 5

Prob. 2 $\hat{\mu}_{LB}$ of $p(\theta) := E[Y_1|\theta]$ based on $X_1, \dots, X_n$?

$Y_1 := \mathbb{1}_{(K,\infty)}(X_1)$, K high threshold

$\theta \sim \Gamma(\gamma, \beta)$

$P(X_1 > x|\theta) = (\lambda/x)^\theta$, $x > \lambda$ (Pareto distr.)

Since $E[X_1] = \infty$, we want to find

$\hat{\mu}_{LB}$ of $p(\theta)$ based on $Y_1, \dots, Y_n$

for $Y_i := \mathbb{1}_{(K,\infty)}(X_i)$

$Y_1, \dots, Y_n$ satisfy the assumptions of Th. 4.2.9

$\longrightarrow$ $\hat{\mu}_{LB} = (1-w)\mu + w\bar{Y}_i$

where $w := \frac{n\lambda}{\varsigma + n\lambda}$ and $\bar{Y}_i := \frac{1}{n}\sum_{j=1}^{n} Y_i$

1. $\mu = E[p(\theta)] = E[\underbrace{P(X_1 > K|\theta)}_{=(\lambda/K)^\theta}]$

$= E[(\lambda/K)^\theta] = \int_0^\infty \underbrace{\left(\frac{\lambda}{K}\right)^y}_{=e^{y\log(\lambda/K)}} \underbrace{e^{-\beta y} y^{\gamma-1} \frac{\beta^\gamma}{\Gamma(\gamma)}}_{f_\theta(y)} \, dy$

$= \underbrace{\left(\int_0^\infty \underbrace{e^{-y(\beta - \log(\lambda/K))} y^{\gamma-1} \frac{\tilde{\beta}^{\tilde{\gamma}}}{\Gamma(\tilde{\gamma})}}_{= f_{\Gamma(\tilde{\gamma},\tilde{\beta})}(y)} dy\right)}_{1=} \frac{\Gamma(\tilde{\gamma})}{\tilde{\beta}^{\tilde{\gamma}}} \cdot \frac{\beta^\gamma}{\Gamma(\gamma)}$

with $\tilde{\gamma} := \gamma$, $\tilde{\beta} := \beta - \log(\lambda/K) \overset{K>\lambda}{>} 0$

$\Rightarrow$ $\mu = \left(\frac{\beta}{\beta - \log(\lambda/K)}\right)^\gamma$

2. $\lambda \overset{\text{def}}{=} Var[p(\theta)] = Var[(\lambda/K)^\theta]$

$= \underbrace{E[(\lambda/K)^{2\theta}]}_{\overset{!}{=} (\beta/\beta - 2\log(\lambda/K))^\gamma} - (\underbrace{E[p(\theta)]}_{=\mu})^2$

$= \left(\frac{\beta}{\beta - 2\log(\lambda/K)}\right)^\gamma - \left(\frac{\beta}{\beta - \log(\lambda/K)}\right)^{2\gamma}$

Exerc. 5

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3. $g \overset{\text{def.}}{=} E[V(\theta)]$

$$V(\theta) \overset{\text{def}}{=} Var[Y_1|\theta] = E[(Y_1 - \overbrace{E[Y_1|\theta]}^{=P(\theta)})^2|\theta]$$

$$\Rightarrow g = E[(Y_1 - P(\theta))^2] = E[\underset{\overset{\shortparallel}{Y_1}}{Y_1^2}] - 2E[(P(\theta))^2] +$$

$$+ E[(P(\theta))^2] = E[P(\theta)] - E[(P(\theta))^2]$$

$$\overset{1,12.}{=} \left(\frac{\beta}{\beta - \log(\lambda/K)}\right)^\gamma - \left(\frac{\beta}{\beta - 2\log(\lambda/K)}\right)^\gamma$$

$\xrightarrow[n=8]{\text{data}}$ $\mu \approx 0.178$, $\lambda \approx 0.66$, $g \approx 0.0801$, $w \approx 0.947$, $Y_i = 0$

$$\Rightarrow \hat{\mu}_{LB} = (1-w)\mu \approx 0.009$$

On the other hand,

$$\hat{\mu}_B = \left(\frac{1 + \log(20) + \log(10)}{1 + \log(20) + \log(10) + \log(100)}\right)^9 \approx 0.007$$

$\longrightarrow$ estimated probability for $X_1 > 100000$ NOK given θ

$\hat{\mu}_{LB}$ is a linear function of the data, whereas $\hat{\mu}_B$ is a non-linear function of those.

<u>risk</u> : $g(\hat{\mu}_{LB}) = (1-w)\lambda \overset{\text{data}}{\approx} 0.035$