Exercises 6

Prob. 1: W.l.o.g. $n=2$

$\longrightarrow \quad Var[Y|\theta] \overset{def}{=} E[(Y-E[Y|\theta])^2]$

$= E[Y^2|\theta] - (E[Y|\theta])^2$

$Y := Y_1 + Y_2$, Y_1, Y_2 indep. given θ

$\Rightarrow \quad Var[Y|\theta] = E[(Y_1+Y_2)^2|\theta] - (E[Y_1|\theta] + E[Y_2|\theta])^2$

$= E[Y_1^2|\theta] - (E[Y_1|\theta])^2 + E[Y_2^2|\theta] - (E[Y_2|\theta])^2$

$+ 2E[Y_1 \cdot Y_2|\theta] - 2E[Y_1|\theta] \cdot E[Y_2|\theta]$ $\quad (*)$

It follows from (4.4) in the lecture notes that

$$E[Y_1 \cdot Y_2|\theta] = \int_{\mathbb{R}^2} y_1 \cdot y_2 \, Q_{Y_1,Y_2}(\omega, dy_1, dy_2)$$

regular conditional probability of Y_1, Y_2 given θ

We know that

$Q_{Y_1,Y_2}(\omega, (-\infty, y_1] \times (-\infty, y_2]) = P(Y_1 \le y_1, Y_2 \le y_2 | \theta)$

$= P(Y_1 \le y_1|\theta) \cdot P(Y_2 \le y_2|\theta) = Q_{Y_1}(\omega, (-\infty, y_1]) \cdot Q_{Y_2}(\omega, (-\infty, y_2])$

for all y_1, y_2 with prob. 1

$\Rightarrow \quad E[Y_1 \cdot Y_2|\theta] = \int_{\mathbb{R}^2} y_1 \cdot y_2 \, Q_{Y_1}(\omega, dy_1) \, Q_{Y_2}(\omega, dy_2)$

$= \int_{\mathbb{R}} y_1 \, Q_{Y_1}(\omega, dy_1) \int_{\mathbb{R}} y_2 \, Q_{Y_2}(\omega, dy_2) \overset{(4.4)}{=} E[Y_1|\theta] \cdot E[Y_2|\theta]$

with prob. 1 $\quad (**)$

$\overset{(*)}{\Longrightarrow} \quad Var[Y_1 + Y_2|\theta] = Var[Y_1|\theta] + Var[Y_2|\theta]$

Choose $Y_i = \frac{1}{2} X_i$, $i = 1, 2$ $\Longrightarrow Var[\bar{X}|\theta] =$

$Var[\frac{1}{2} X_1|\theta] + Var[\frac{1}{2} X_2|\theta] = \frac{1}{4} Var[X_1|\theta] + \frac{1}{4} Var[X_2|\theta]$

$\overset{\text{Def 4.2.5(v)}}{=} 2 \cdot \frac{1}{4} Var[X_1|\theta] = \frac{1}{2} Var[X_1|\theta]$

$\Rightarrow \quad E[Var[\bar{X}|\theta]] = \frac{1}{2} E[Var[X_1|\theta]]$

④

Exerc. 6

Prob. 4: (i) $X_i \overset{\text{def}}{=} \max(Y_{i-1}, Y_i)$, $Y_i, i \geq 0$ i.i.d. with common distr. function $F^* \overset{\text{def}}{=} \sqrt{F}$, F continuous distr. funct.

Recall: F distr. function of a r.v. $\Longleftrightarrow$

1) F non-decreasing
2) F right-continuous
3) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

$\Longrightarrow F^* \overset{\text{def}}{=} \sqrt{F}$ distr. function of a r.v.

Recall: F continuous distr. funct. $\Longleftrightarrow$

$$F(x) = \int_{-\infty}^{x} f(y)\,dy, \; x \in \mathbb{R} \text{ for some } f \geq 0$$

$$\Longrightarrow P(Y_0 < x) = P(Y_0 \leq x) \qquad (*)$$

Let $k \geq 2$

$\longrightarrow X_i = \max(Y_{i-1}, Y_i)$, $X_{i+k} = \max(Y_{i+k-1}, Y_{i+k})$ indep.

$\Longrightarrow X_i$ indep. of X_{i+k} for $k \geq 2$, $i \geq 1$

Let $k = 1$

$$\longrightarrow P(X_i \leq x, X_{i+1} \leq x) = P(Y_{i-1} \leq x, Y_i \leq x, Y_i \leq x, Y_{i+1} \leq x)$$

$$\underset{\text{i.i.d.}}{\overset{Y_i, i\geq 0}{=}} P(Y_0 \leq x)^3 = (F^*(x))^3 \qquad (**)$$

On the other hand,

$$P(X_i \leq x) \cdot P(X_{i+1} \leq x) = (F^*(x))^2 \cdot (F^*(x))^2 = (F^*(x))^4$$

Suppose that X_i, X_{i+1} indep.

$\overset{(**)}{\Longrightarrow} (F^*(x))^3 = (F^*(x))^4 \Longrightarrow F^*(x) \in \{0,1\}$ for all x

$\underset{\text{cont. in } x}{\overset{F^*(x)}{\Longrightarrow}} F^* \equiv 0$ or $1 \Longrightarrow$ contradiction.

$\Longrightarrow X_i, X_{i+1}$ dependent

(5) Exerc. 6

Prob. 4 : (ii) $P(X_i \le x) = P(Y_{i-1} \le x, Y_i \le x) \overset{Y_i, i \ge 0}{\underset{\text{i.i.d}}{=}}$

$= P(Y_{i-1} \le x) P(Y_i \le x) = P(Y_0 \le x) P(Y_0 \le x) = (F^*(x))^2 = F(x)$

for all x

(iii) Let $A \in \mathcal{B}(E)$, $E = \mathbb{R}$ (Borel sets) $\Longrightarrow$

$N_n(A) \overset{\text{def}}{=} \sum_{i=1}^{n} \varepsilon_{X_i}(A) = \sum_{i=1}^{n} \mathbb{1}_A(X_i)$ (r.v.)

$\overset{\text{Def. 5.13}}{\Longrightarrow}$ N_n point proc.

We want to show that N_n is non-simple point proc.

$\longrightarrow$ sufficient to show that $X_i = X_{i+1}$ for a positive prob. for some $i \ge 1$

We see that $X_i = X_{i+1} \Longleftrightarrow \max(Y_{i-1}, Y_i) = \max(Y_i, Y_{i+1})$

if e.g. $Y_{i-1} \le Y_i$ and $Y_i \ge Y_{i+1}$

Y_{i-1}, Y_{i+1} indep of Y_i

$P(Y_{i-1} \le Y_i, Y_{i+1} \le Y_i) = E[\mathbb{1}_{\{Y_{i-1} \le Y_i, Y_{i+1} \le Y_i\}}]$

$\overset{\text{dobbel-forventning}}{=} E\Big[\big(\underbrace{E[\mathbb{1}_{\{Y_{i-1} \le x, Y_{i+1} \le x\}}]}_{\substack{= P(Y_{i-1} \le x, Y_{i+1} \le x) \\ \overset{Y_i, i \ge 0 \text{ i.i.d.}}{=} (P(Y_0 \le x))^2 = (F^*(x))^2}}\big)\Big|_{x = Y_i}\Big]$

$= E[(F^*(Y_0))^2]$ uniform distr. (***)

$F^*(Y_0) \sim U(0,1)$

e.g. if F^* has an inverse $(F^*)^{-1}$, we see that

$P(F^*(Y_0) \le x) = P(Y_0 \le (F^*)^{-1}(x)) \overset{Y_0 \sim F^*}{=} F^*((F^*)^{-1}(x)) = x$

$\Longrightarrow$ the r.v. $F^*(Y_0)$ is uniformly distr. on $(0,1)$

$\Longrightarrow$ $F^*(Y_0)$ has a prob. density $f \equiv 1$

$\overset{(***)}{\Longrightarrow}$ $E[(F^*(Y_0))^2] = \int_0^1 y^2 \cdot \underbrace{f(y)}_{=1}\, dy = \frac{1}{3}$

$\overset{(***)}{\Longrightarrow}$ $P(X_i = X_{i+1}) \ge \frac{1}{3}$

$\Longrightarrow$ N_n non-simple point proc.