②

<u>Exerc. 6</u>

<u>Prob. 2</u>: Choose a Poisson random measure N with mean measure $\mu(A) := \tilde{\lambda}\lambda(A)$, $\tilde{\lambda} > 0$, $A \in \mathcal{B}([0,\infty))$, λ Lebesgue-Borel measure, $E = [0,\infty)$

Define the process

$$N(t) = N([0,t]) < \infty, \quad t \geq 0$$

Let $0 = t_0 < t_1 < \ldots < t_n$

$$\longrightarrow \quad N(t_1) - \underbrace{N(t_0)}_{=0} = N((0,t_1]), \ldots, N(t_n) - N(t_{n-1}) = N((t_{n-1}, t_n])$$

indep. r.v.'s, since $(0,t_1], (t_1,t_2], \ldots, (t_{n-1},t_n]$ are disjoint (see Def. 5.19 (ii))

On the other hand,

$$N(t+h) - N(t) = N((t,t+h]) \overset{\text{Def. 5.19(i)}}{\sim} \text{Pois}(\underbrace{\mu((t,t+h])}_{=\tilde{\lambda}\cdot h}), \quad h \geq 0$$

Poisson distr.

$$N(t+\tfrac{1}{n}) = N([0,t+\tfrac{1}{n}]) =$$

$$= \int_E \underbrace{\mathbb{1}_{[0,t+\frac{1}{n}]}(s)}_{\xrightarrow[n\to\infty]{} \mathbb{1}_{[0,t]}(s)} N(\omega,ds) \xrightarrow[n\to\infty]{\text{dom. conv.}} \int_E \mathbb{1}_{[0,t]}(s)\, N(\omega,ds)$$

$$= N([0,t]) = N(t) \implies N(t),\ t \geq 0 \text{ right-cont. with prob. 1}$$

$N(t) = N([0,t])$ non-decreasing in t $\implies$ left limits exist with prob. 1

$\overset{\text{Rem. 3.1.2}}{\implies}$ $N(t), t \geq 0$ Poisson proc. with intensity $\tilde{\lambda}$

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Exerc. 6

Prob. 3 : N_1, N_2 point processes

"$\Leftarrow$" : $\Psi_{N_1}(g) = E[\exp(-\int_E g(z) N_1(\omega, dz))] =$

$= E[\exp(-\int_E g(z) N_2(\omega, dz))] = \Psi_{N_2}(g)$

for all bounded measurable $g \geq 0$

Choose $g(z) = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}(z)$, $A_i \in \mathcal{B}(E)$ (Borel sets), $\lambda_i \geq 0$, $i = 1, \dots, n$

$\Rightarrow$ $E[\exp(-\sum_{i=1}^n \lambda_i \underbrace{N_1(\omega, A_i)}_{= \int_E \mathbb{1}_{A_i}(z) N_1(\omega, dz)})] = E[\exp(-\int_E \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}(z) N_1(\omega, dz))]$

$= E[\exp(-\int_E \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}(z) N_2(\omega, dz))] =$

$= E[\exp(-\sum_{i=1}^n \lambda_i N_2(\omega, A_i))]$ for all $\lambda_1, \dots, \lambda_n \geq 0$ $\quad (*)$

Recall : $X = (X_1, \dots, X_n)'$, $X_i \geq 0$, $i = 1, \dots, n$ r.v.'s

$\longrightarrow$ Laplace-Stieltjes transform φ_X of X defined by

$\varphi_X(\lambda_1, \dots, \lambda_n) = E[\exp(-\sum_{i=1}^n \lambda_i X_i)]$

$\longrightarrow$ $X \overset{d}{=} Y = (\underset{\geq 0}{Y_1}, \dots, \underset{\geq 0}{Y_n})' \Longleftrightarrow \varphi_X(\lambda_1, \dots, \lambda_n) =$

$\varphi_Y(\lambda_1, \dots, \lambda_n)$ for all $\lambda_1, \dots, \lambda_n \geq 0$ $\quad (**)$

$\overset{(*)}{\Longrightarrow}$ $(N_1(A_1), \dots, N_1(A_n))' \overset{d}{=} (N_2(A_1), \dots, N_2(A_n))'$ $\quad (***)$

for all $A_1, \dots, A_n \in \mathcal{B}(E)$, $n \geq 1$ $\overset{\text{Def. 5.15}}{\Longrightarrow}$ $N_1 \overset{d}{=} N_2$

"$\Rightarrow$" $N_1 \overset{d}{=} N_2 \Longleftrightarrow (***) \overset{(**)}{\Longleftrightarrow}$

$\Psi_{N_1}(g) = E[-\int_E g(z) N_1(\omega, dz)] = E[\exp(-\sum_{i=1}^n \lambda_i N_1(\omega, A_i))]$

$= E[\exp(-\sum_{i=1}^n \lambda_i N_2(\omega, A_i))] = E[\exp(-\int_E g(z) N_2(\omega, dz))]$

$= \Psi_{N_2}(g)$ for $g(z) := \sum_{i=1}^n \underset{\geq 0}{\lambda_i} \mathbb{1}_{A_i}(z)$ step function

$\xrightarrow{\text{approximation}}$ general case