1

Non-Life Insurance Mathematics

1. Introduction

Basic literature

1. Mikosch, Thomas: Non-Life Insurance Mathematics. Springer, 2nd edition (2009).
2. Bølviken, Erik: Computation and Modelling in Insurance and Finance. Cambridge Univ. Press (2014).

What is the objective of general insurance?

General insurance offers protection against risk | of future events which could negatively affect ... | that a future event will unexpectedly diminish the normal quality of life of the insured and his dependants.

Ex.: Quality of life could be impaired tomorrow as a result of a motor cycle accident

→ General insurance provides compensation for such events in form of a fixed amount of money specified by an

→ insurance contract: A legally binding contract of economic character between the insurer (e.g. company or syndicate) and the insured (policyholder). When entering into the contract the insured pays premiums. In return the insurance company will provide e.g. the replacement value of a loss (erstatningsbeløp)

$X(\omega)$,

if an "event"

$\omega \in \Omega$

occurs during the contract period $[0,T]$, $T \leq \infty$

$\longrightarrow$ Ω collection of all possible outcomes
$\Longrightarrow$ outcome of X uncertain
$\longrightarrow$ random variable

In practice, large groups of policyholders contribute to collective funds to bear individual losses more easily

Actuarial science distinguishes between
Life insurance and Non-Life insurance

1. Life insurance (LI) provides financial service to ensure policyholders and their families against financial losses caused e.g. by
   - death
   - sickness or disability

   or to pay benefits in the case of survival
2. Non-Life insurance (NLI) is somewhat the contrary to LI and deals e.g. with property or liability insurance (ansvarsforsikring)

$\longrightarrow$ However, health insurance is in the overlap of these domains
$\longrightarrow$ no clear separation between LI and NLI

$\longrightarrow$ differences:
(i) non-life policies are more short-term and less predictable in size than life insurance policies
(ii) evaluation and identification of factors influencing the degree of risk are more difficult in NLI than in LI
(iii) range of events giving rise to a claim payment is in NLI broader than in LI
(iv) measurement of risk exposure is more difficult in NLI than in LI

$\longrightarrow$ Central problems in classical (Non)-Life insurance:

③ 1. Premium calculation:

What are "fair" insurance premiums?

Consider e.g. the total loss L of the insurer given by

$$L = Z - P,$$

where

$Z \stackrel{\text{def}}{=}$ PV (present value) of the benefits payable by the insurer (random variable)

and $P \stackrel{\text{def}}{=}$ PV of the premiums provided by the insured

A calculation method which also takes into account the risk preferences of the insurer is e.g. the utility-based equivalence principle:

Here premiums are calculated by requiring that

$$E[u(-L)] = u(0), \tag{1.1}$$

where

$u(x)$ utility function, that is $u'(x) > 0$ and $u''(x) < 0$ for all x

$\longrightarrow$ $u(x)$ measures the utility of the amount of money x from the viewpoint of the insurer

$\longrightarrow$ Interpretation of (1.1):

The expected utility of the insurer's gain (i.e. $x = -L$) should be equal to the utility of the "gain" zero

$\longrightarrow$ (1.1) provides a "fairness condition" for premium calculation

2. Premium reserves:

Consider now the total loss of the insurer at at future time $t \geq 0$, that is

$${}_tL \stackrel{\text{def}}{=} {}_tZ - {}_tP, \tag{1.2}$$

where

${}_tZ$ PV of the future benefits at time t

and ${}_tP$ PV " premiums "

(4) Which amount of money should a (non-) life company <u>reserve</u> to meet its liabilities (i.e. ${}_tL$) ?

→ <u>possible solution</u> : (net) premium reserve ${}_tV$ at time t is defined as

$$ {}_tV = E[{}_tL \mid T > t], $$

T e.g. future life time of a life or the first arrival time of a claim

→ It is reasonable for both the insurer and the policyholder to require the following "<u>fairness conditions</u>":

1. $E[L] = 0$ (equivalence principle)
2. ${}_tV \geq 0$ for all $t \geq 0$ | i.e. to make the contract interesting for the policyholder

3. <u>Distribution of the total claim amount in an insurance portfolio</u>:

Denote by

$$ X_h $$

the claim caused by insurance policy h $(h = 1, \ldots, n)$ during a given period of time (e.g. 1 year)

→ <u>total (or aggregate) amount of claims</u>:

$$ S = \sum_{h=1}^{n} X_h $$

→ <u>Problems</u>:

What are appropriate stochastic models for $X_1, \ldots, X_n$?

How can we compute or approximate $P(S \leq x)$, that is the probability that $S \leq x$?

4. <u>Reinsurance</u>:

Consider e.g. a portfolio of insurance policies with high risk for large total claim amounts S. <u>In order</u> to reduce the probabilities for large values of S the insurance buys a <u>reinsurance</u>

5

with reimbursement of the amount R

→ R function of the individual claims

→ R random variable

→ "new" total claim amount : retention

$$\hat{S} = S + \Pi - R$$

Π premium for the reinsurance

→ idea of reinsurance :

$$P(\hat{S} \geq x) < P(S \geq x)$$

for large x

→ reduction of probabilities for large values of S

Modern (non-) life insurance also deals with equity-linked insurance

→ equity-linked insurance contracts (investment guarantees) allow the policyholder participation in an underlying stock index or fund

Ex. of an investment guarantee :

Guaranteed Minimum Maturity Benefit (GMMB)

policyholder gets the payoff

$$\max(G, F_T)$$

G guaranteed minimum return

F_T value of a fund at terminal time T

→ insurer's liability is given by

$$\max(G - F_T, 0)$$

→ payoff of a put option with strike price G

Central problem :

What is the fair value of an investment guarantee (taking into account the insurer's risk preferences) ?

→ possible solution method : dynamical hedging principle or risk indifference pricing from finance

⑥ In this course we want to discuss the following crucial issues of non-life insurance mathematics:

1. Ruin theory:

In this area of insurance mathematics one aims at studying the impact of both small claims and large claims on risk theory models

In this context one is e.g. interested in the calculation or estimation of the probability for ruin of an insurance portfolio, the so-called ruin probability, which can be considered a measure for risk in an insurance portfolio.

To explain this concept in more detail, let $S(t)$ be the total claim amount at time t, that is

$$S(t) \overset{def.}{=} \sum_{i=1}^{N(t)} X_i ,$$

where

(i) $X_1, X_2, X_3, \ldots$ are the claim sizes (e.g. fire losses)

and (ii) $N(t)$ is the number of claims arriving by time t

Further, consider insurer's surplus $U(t)$ at time t, that is

$$U(t) \overset{def.}{=} u + p(t) - S(t) ,$$

where

(i) $u > 0$ initial capital

and (ii) $p(t)$ premium income at time t

→ The event
$$\text{Ruin} \overset{\text{def}}{=} \{U(t) < 0 \text{ for some } t > 0\}$$
is called ruin
→ event that U ever falls below 0

Then the <u>ruin probability</u> $\psi(u)$ is defined as
$$\psi(u) = P(\text{Ruin} \mid U(0) = u)$$

→ a <u>central goal</u> in insurance business is to keep $\psi(u)$ small for sufficiently large initial capital u

2. Experience rating:

Experience rating is concerned with the analysis of the <u>different types of risk</u> represented by claims in an insurance portfolio; as time goes by

→ risks are heterogeneous in a portfolio
→ reasonable to introduce <u>individual models</u> for policyholders based on their claim history or category (e.g. driving skill and experience or age of the driver) in connection with calculation of premiums or the distribution of total claim amounts

→ <u>possible method</u>: <u>credibility theory</u>
→ <u>Bayes estimation</u> based on available data of individual policies

3. Cluster point process model:

The cluster point process model, which contains the famous <u>chain ladder model of Mack</u> as a special case, is a <u>popular</u> model in insurance and is used to <u>forecast future claim numbers</u> and <u>total claim amounts given</u> the claim number and total claim amount history

⑧ In order to study this model, we also have to discuss the concept of point processes and their weak convergence, which generalize e.g. the notion of a Poisson process.