(49) in the same way as for prob. meas. P (see Def. 2.9).

Ex.: expectation $E[X]$ or $\int_0^T f(s)\, \underbrace{ds}_{=\lambda(ds)}$ Riemann-integral ← Lebesgue-Borel-meas., $\Omega=[0,T]$

Def. 5.7 (metric space (E,d))

(E,d) is called metric space, if d is a metric on $E \neq \emptyset$ that is $d: E\times E \longrightarrow [0,\infty)$ s.t.

(i) $d(x,y)=d(y,x)$ (ii) $d(x,z) \le d(x,y) + d(y,z)$ (triangle inequality) and

distance between y,z

(iii) $d(x,y)=0 \Longleftrightarrow x=y$

for all $x,y,z\in E$

Ex.: $E=\mathbb{R}^d$, $d(x,y):=\|x-y\| = \left(\sum_{i=1}^d (x_i-y_i)^2\right)^{1/2}$, $x=(x_1,\dots,x_d)'$, $y=(y_1,\dots,y_d)'$

Def. 5.8 (complete separable metric space (E,d))

(i) (E,d) complete, if for all $x_n, n\geq 1$ in E with Cauchy sequence

$d(x_n,x_m) \xrightarrow[n,m\to\infty]{} 0$ there exists a $x\in E$ s.t. $d(x_n,x)\xrightarrow[n\to\infty]{} 0$

(ii) (E,d) separable, if there ex. a countable set A $(=\{y_1,y_2,\dots\})$ in E st. for all x there ex. $z_1,z_2,\dots$ in A s.t.

$$d(z_n,x)\xrightarrow[n\to\infty]{} 0$$

Ex. $(\mathbb{R}^d, \|\cdot\|)$ Euclidean space

Def. 5.9 (Compactness in (E,d))

$A\subseteq E$ compact $\Longleftrightarrow$ every sequence $x_n, n\geq 1$ in A has a convergent subsequence in A.

Ex.: $A\subseteq E=\mathbb{R}^d$ compact $\Longleftrightarrow$ A bounded and closed

In the sequel we shall assume that (E,d) is a complete separable metric space (or $E\subseteq\mathbb{R}^d$ Borel set) and denote by $\mathcal{B}(E)$ the Borel-σ-algebra on E, that is

$$\mathcal{B}(E) = \text{smallest } \sigma\text{-algebra containing all open balls } U_\varepsilon(x) := \{y\in E : d(y,x)<\varepsilon\},\ \varepsilon>0,\ x\in E \quad (5.1)$$

→ important class of measures on $\mathcal{B}(E)$ w.r.t. applications

Def. 5.10 (Point measures)

Let $\mathcal{A}^* = \mathcal{B}(E)$ and denote by ε_x the Dirac measure at $x\in E$ defined as (see Section 1)

$$\varepsilon_x(A) = \begin{cases} 1, & \text{if } x \in A \\ 0 & \text{else} \end{cases}$$

Then a measure m on $\mathcal{A}^*$ is called <u>point measure</u>, if there exists a sequence $x_i, i \geq 1$ in E s.t.

(i) $$m(A) = \sum_{i \geq 1} \varepsilon_{x_i}(A)$$

for all $A \in \mathcal{A}^*$ and

(ii) $m(K) < \infty$ for all compact sets $K \subseteq E$

The class of all point measures is denoted by $M_p(E)$.

<u>Rem. 5.11</u>: (i) $m(A) = \sum_{i \geq 1} \varepsilon_i(A) = \sum_{i \geq 1} \mathbb{1}_A(x_i) =$

number of $= \#\{i \geq 1 : x_i \in A\} \longrightarrow m$ <u>counting measure</u>

(ii) $m(K) < \infty$, K compact $\longrightarrow$ K cannot contain infinitely many x_i

(iii) $f \geq 0$ $\mathcal{A}^*$-measurable, $m(A) = \sum_{i \geq 1} \varepsilon_{x_i}(A)$, $A \in \mathcal{A}^*$

$$\int_E f(y)\, m(dy) = \sum_{i \geq 1} f(x_i) \qquad (5.2)$$

<u>Def. 5.12</u> (Simple point measures)

Define $n_i = \#\{j \geq 1 : x_i = x_j\}$ (multiplicity of x_i).

Let $m(A) = \sum_{i \geq 1} \varepsilon_{x_i}(A)$, $A \in \mathcal{A}^*$ be a point measure.

If $n_i = 1$ for all $i \geq 1$ then m is called <u>simple point measure</u>.

<u>Rem</u>: $x_i \neq x_j, i \neq j \longrightarrow m(A) = \sum_{i \geq 1} \varepsilon_{x_i}(A), A \in \mathcal{A}^*$ simple point meas

$\longrightarrow$ <u>Def. 5.13</u> (Point process)

A function $N : \Omega \longrightarrow M_p(E)$ is said to be a <u>point process</u>, if $N(\omega, A)$ $(\in \{0, 1, \dots, \infty\})$ is a <u>random variable</u> for all $A \in \mathcal{A}^* (= \mathcal{B}(E))$.

It is called <u>simple point process</u> if $N(\omega, A)$ is a simple point measure in A with prob. 1.

<u>Ex 5.14</u>: $E = \mathbb{R}^d$, $X_i : \Omega \longrightarrow E$ random vector, $i \geq 1$ s.t. $X_i(\omega) \in B \in \mathcal{A}^*$ for finitely many i, if B bounded, $\omega \in \Omega$

(51) $\longrightarrow$ $N(\omega, A) := \sum_{i \geq 1} \varepsilon_{X_i(\omega)}(A) = \sum_{i\geq 1} 1_A(X_i(\omega))$

$= \# \{ i \geq 1 : X_i(\omega) \in A \}$ defines a point process.

If $X_i \neq X_j$, $i \neq j$ with prob. 1 $\longrightarrow$ $N(\omega, A)$ simple point proc.

<u>Def. 5.15</u> ($N_1 \stackrel{d}{=} N_2$)

We shall say that two point processes N_1 and N_2 are <u>equal in distribution</u>, if for all m, $A_1, \dots, A_m \in \mathcal{E}^*$ the rand. vectors $Y_1 := (N_1(A_1), \dots, N_1(A_m))'$ ← transpose and $Y_2 := (N_2(A_1), \dots, N_2(A_m))'$ have the same distribution. In this case we write $N_1 \stackrel{d}{=} N_2$.

<u>Rem. 5.16</u>: (i) Define $\mathcal{B}$:= smallest σ-alg. on $\Omega := M_p(E)$ containing all sets $\{ m \in M_p(E) : m(A) \in C \}$, $A \in \mathcal{B}(E)$, $C \subseteq [0, \infty]$.

Consider $N : \Omega \longrightarrow M_p(E)$ a rand. variable w.r.t. $\mathcal{B}$, i.e.

$$\{ \omega \in \Omega : N(\omega) \in G \} \in \mathcal{A} \qquad (*)$$

for all $G \in \mathcal{B}$ $\longrightarrow$ $(*)$ $\Longleftrightarrow$ Def. 5.13

(ii) $N_1 \stackrel{d}{=} N_2 \Longleftrightarrow P(N_1 \in G) = P(N_2 \in G)$ for all $G \in \mathcal{B}$

$\longrightarrow$ Examples of <u>point processes in insurance applications</u>:

<u>Ex. 5.17</u> (Renewal process)

$T_i := W_1 + \dots + W_i$, W_i i.i.d. and positive, W_i i-th inter-arrival time. Define $N : \Omega \longrightarrow M_p(E)$ for $E = (0, \infty)$ by

$$N(\omega, A) = \sum_{i\geq 1} \varepsilon_{T_i}(A) = \sum_{i \geq 1} 1_A(T_i) = \# \{ i \geq 1 : T_i \in A \}$$

$\longrightarrow$ <u>simple</u> point process, since $T_i \neq T_j$, $i \neq j$ with prob. 1 and $T_i \xrightarrow[i\to\infty]{} \infty$ with prob. 1 by the SLLN $\Longrightarrow$ $N(\omega, K) < \infty$, $K \subseteq E$ compact with prob. 1

$\xrightarrow{A = (0,t]}$ $N(\omega, A) = N(t)$, $t \geq 1$ renewal proc. as a special case

<u>Ex. 5.18</u> (Renewal model)

$T_i := W_1 + \dots + W_i$, $i \geq 1$, $W_i > 0$, i.i.d., T_i, $i \geq 1$ indep. of i.i.d. claim sizes $\longrightarrow$ $N : \Omega \longrightarrow M_p(E)$ for $E = (0,\infty) \times (0, \infty)$ given by

$$N(\omega, A) = \sum_{i \geq 1} \varepsilon_{(T_i(\omega), X_i(\omega))}(A) = \# \{ i \geq 1 : (T_i(\omega), X_i(\omega)) \in A \}$$

is a simple point process