(57) Rem.: Recall: $X_n, n \ge 1, \xi$ r.v.'s. Then

$$X_n \xrightarrow[n\to\infty]{d} \xi \overset{def}{\Longleftrightarrow} F_{X_n}(x) \longrightarrow F_\xi(x) \overset{def}{=} P(\xi \le x) \quad (5.18)$$

for all continuity points x of the function F_ξ

Proof of Prop. 5.24:

(i) follows directly from (5.12)

(ii) Idea: central limit theorem (CLT)

We see that

$$\sqrt{m}\,[\hat f_j^{(m)} - f_j] \overset{def}{=} \sqrt{m}\left[\frac{\bar N_m^{(j+1)}}{\bar N_{m-1}^{(j)}} - f_j\right] = \frac{\sqrt{m}}{\bar N_{m-1}^{(j)}}\left(\bar N_m^{(j+1)} - f_j \bar N_{m-1}^{(j)}\right)$$

$$= \underbrace{\frac{(m/(m-j-1))^{1/2}}{\bar N_{m-1}^{(j)}}}_{=:Y_m} \underbrace{\left(\frac{1}{\sqrt{m-j-1}}\sum_{i=1}^{m-j-1}(N_{i,i+j+1} - f_j N_{i,i+j})\right)}_{=:Z_m} \quad (*)$$

Recall CLT: $X_i, i \ge 1$ i.i.d. with $E[X_1]=\mu$, $Var[X_1]=\sigma^2<\infty$

$$\Longrightarrow \sqrt{n}\,(\bar X_n - \mu) \xrightarrow[n\to\infty]{d} \xi, \quad \xi \sim N(0,\sigma^2) \quad (5.19)$$

$\frac{1}{n}\sum_{i=1}^{n} X_i$

Choose $X_i = N_{i,i+j+1} - f_j N_{i,i+j}$ i.i.d. with $E[X_1] = \mu \overset{(5.10)}{=} 0$

$$\overset{(5.19)}{\Longrightarrow} Z_m \xrightarrow[m\to\infty]{d} \hat\xi, \quad \hat\xi \sim N(0,\hat\sigma^2), \quad \hat\sigma^2 = Var[X_1] \quad (**)$$

Recall: $X_n \xrightarrow[n\to\infty]{d} X$, $Y_n \xrightarrow[n\to\infty]{} c$ constant with prob. 1

$$\Longrightarrow Y_n \cdot X_n \xrightarrow[n\to\infty]{d} c \cdot X \quad (5.20)$$

$Y_m \xrightarrow[m\to\infty]{} \frac{1}{E[N_{1,1+j}]}$ with prob. 1 $\overset{(5.20)\text{ for } X_m := Z_m,\ Y_m := Y_m,\ (*)}{\Longrightarrow}$

$$\sqrt{m}\,[\hat f_j^{(m)} - f_j] = Y_m \cdot X_m \xrightarrow[m\to\infty]{d} \xi := \frac{1}{E[N_{1,1+j}]} \cdot \hat\xi, \quad \hat\xi \sim N(0,\hat\sigma^2)$$

$$\Longrightarrow \xi \sim N\left(0, \frac{Var[X_1]}{(E[N_{1,1+j}])^2}\right) \quad (***)$$

$Var[X_1] \overset{def}{=} Var[N_{1,1+j+1} - f_j N_{1,1+j}] = Var[N_{1,1+j+1}] + f_j^2 Var[N_{1,1+j}] - 2 f_j Cov[N_{1,1+j}, N_{1,1+j+1}]$

Further, $Cov[N_{1,1+j}, N_{1,1+j+1}] = E[N_{1,1+j+1} \cdot N_{1,j+1}] - E[N_{1,1+j+1}] \cdot E[N_{1,1+j}] = E[N_{1,1+j}\, E[N_{1,1+j+1} \mid N_{1,1+j}]] - E[E[N_{1,1+j+1} \mid N_{1,1+j}]]\, E[N_{1,1+j}] \overset{(5.10)}{=} f_j Var[N_{1,1+j}]$

and $Var[N_{1,1+j+1}] \overset{(5.18)}{=} E[(N_{1,1+j+1})^2] - f_j^2 E[(N_{1,1+j})^2]$

$$\Longrightarrow \frac{Var[X_1]}{(E[N_{1,1+j}])^2} = \left(N[(N_{1,1+j+1})^2] - f_j^2 E[(N_{1,1+j})^2]\right) / (E[N_{1,1+j}])^2$$

$\Longrightarrow$ proof.

(58) <u>Rem.</u>: It follows directly from the SLLN that the asymptotic variance σ^2 in (5.17) can be approximated by the strongly consistent estimator

$$\hat{\sigma}_m^2 := \frac{\sum_{i=1}^{m-j-1}\left((N_{i,i+j+1})^2 - (\hat{f}_j^{(m)})^2 (N_{i,i+j})^2\right)}{(m-j)^{-1}\left(\sum_{i=1}^{m-j} N_{i,i+j}\right)^2} \qquad (5.21)$$

<u>Lemma 5.25</u> ($E[\hat{f}_j^{(m)}]$, $Cov[\hat{f}_{j_1}^{(m)}, \hat{f}_{j_2}^{(m)}]$)

Consider Mack's model. Then

(i) The chain ladder estimator is <u>unbiased</u>, that is

$$E[\hat{f}_j^{(m)}] = f_j, \; j = 0, \dots, m-2$$

(ii) If $Var[N_{1,1+j}] < \infty$, $j \geq 0$, then the chain ladder estimators are <u>uncorrelated</u>, that is

$$Cov[\hat{f}_{j_1}^{(m)}, \hat{f}_{j_2}^{(m)}] = 0, \; 0 \leq j_1 < j_2 \leq m-2$$

<u>Proof</u>: Define

$$\mathcal{G}_j^{(m)} = \sigma(N_{i,i+k}, 0 \leq k \leq j, 1 \leq i+k \leq m) = \text{smallest } \sigma\text{-alg.}$$

by the events $\{N_{i,i+k} \in A_{i,i+k}\}$, $A_{i,i+k} \subseteq [0,\infty)$, $0 \leq k \leq j$, $1 \leq i+k \leq m$

$$= \text{"} N_{i,i+k}, 0 \leq k \leq j, 1 \leq i+k \leq m \text{"} \qquad (5.22)$$

and $\mathcal{F}_{i,i+k} = \sigma(N_{i,i}, \dots, N_{i,i+k}) = \text{"} N_{i,i}, \dots, N_{i,i+k} \text{"}$, similarly

(i) <u>Recall</u>: X, Y r.v., Y r.v. w.r.t. $\mathcal{A} \subseteq \mathcal{F}$. Then

$$E[Y \cdot X | \mathcal{A}] = Y \cdot E[X|\mathcal{A}] \qquad (5.23)$$

If $\mathcal{A} = \mathcal{G}_j^{(m)}$, then Y r.v. w.r.t $\mathcal{G}_j^{(m)} \iff$ $(*)$

Y is a (measurable) function of $N_{i,i+k}$, $0 \leq k \leq j$, $1 \leq i+k \leq m$

We know that

$$E[\hat{f}_j^{(m)} | \mathcal{G}_j^{(m)}] \overset{\text{def}}{=} E\Big[\underbrace{\frac{1}{\sum_{i=1}^{m-j-1} N_{i,i+j}}}_{=Y} \cdot \underbrace{\Big(\sum_{i=1}^{m-j-1} N_{i,i+j+1}\Big)}_{=X} \Big| \mathcal{G}_j^{(m)}\Big]$$

$$\overset{(5.23),(*)}{=} \frac{\sum_{i=1}^{m-j-1} E[N_{i,i+j+1} | \mathcal{G}_j^{(m)}]}{\sum_{i=1}^{m-j-1} N_{i,i+j}} \qquad (**)$$

(59) <u>Recall</u> from the proof of Exerc. 4, Prob. 2:

index set

X r.v. indep. of r.v.'s $Y_i, i \in J_1$. Let $X_i, i\in J_2$ r.v.'s. Then

$$E[X \mid X_i, i\in J_2, Y_i, i \in J_1] = E[X \mid X_i, i \in J_2] \qquad (5.24)$$

$\longrightarrow$ Choose $X = N_{i,i+j+1}$, $Y_{l,k} = N_{l,l+k}$, $k \geq 0$, $l \neq i$,

$X_k = N_{i,i+k}$, $k = 1,\dots,j$ indep. due to (iii) in Def. 5.22 $\quad i+j \leq m$

$$\Rightarrow E[N_{i,i+j+1} \mid \mathcal{G}_j^{(m)}] \overset{def}{=} E[N_{i,i+j+1} \mid Y_{l,k}, 0\leq k \leq j, 1 \leq l+k \leq m, l \neq i, X_k, k=1,\dots,j]$$

$$\overset{(5.24)}{=} E[N_{i,i+j+1} \mid \underbrace{X_k, k=1,\dots,j}_{=\mathcal{F}_{i,i+j}}] \overset{(5.10)}{=} f_j \cdot N_{i,i+j}$$

$$\overset{(**)}{\Longrightarrow} E[\hat{f}_j^{(m)} \mid \mathcal{G}_j^{(m)}] = f_j \qquad (5.25)$$

$E[\cdot]$ on both sides (i).

(ii) Let $j_1 < j_2$. Then $\hat{f}_{j_1}^{(m)}$ is a function of the r.v.'s $N_{l,l+k}$, $0 \leq k \leq j_2$, $1 \leq l+k \leq m$

$$\overset{(5.23),(*)}{\Longrightarrow} E[\hat{f}_{j_1}^{(m)} \hat{f}_{j_2}^{(m)} \mid \mathcal{G}_{j_2}^{(m)}] = \hat{f}_{j_1}^{(m)} E[\hat{f}_{j_2}^{(m)} \mid \mathcal{G}_{j_2}^{(m)}] \overset{(i)}{=} \hat{f}_{j_1}^{(m)} \cdot f_{j_2}$$

$$\xRightarrow{E[\,]\text{ on both sides}} Cov[\hat{f}_{j_1}^{(m)}, \hat{f}_{j_2}^{(m)}] \overset{def}{=} E[\hat{f}_{j_1}^{(m)} \hat{f}_{j_2}^{(m)}] - E[\hat{f}_{j_1}^{(m)}] \cdot E[\hat{f}_{j_2}^{(m)}]$$

$$= E[\underbrace{\hat{f}_{j_1}^{(m)} f_{j_2}}_{=f_{j_1} f_{j_2}}] - f_1 \cdot f_2 = 0 \Longrightarrow \text{(ii)} \Longrightarrow \text{proof.}$$

<u>Next aim</u>: <u>Calculation of $Var[\hat{f}_j^{(m)}]$</u>:

$$\hat{N}_{i,i+j} := N_{i,i+j+1} - f_j N_{i,i+j}$$

$$\Rightarrow Var[\hat{f}_j^{(m)} \mid \mathcal{G}_j^{(m)}] \overset{def}{=} E[(\hat{f}_j^{(m)} - \overbrace{E[\hat{f}_j^{(m)} \mid \mathcal{G}_j^{(m)}]}^{\overset{(5.25)}{=} f_j})^2 \mid \mathcal{G}_j^{(m)}]$$

$$= E\left[\left(\frac{\sum_{i=1}^{m-j-1} \hat{N}_{i,i+j}}{\sum_{i=1}^{m-j-1} N_{i,i+j}}\right)^2 \Bigm| \mathcal{G}_j^{(m)}\right] \overset{(5.23),(*)}{=} \frac{1}{\left(\sum_{i=1}^{m-j-1} N_{i,i+j}\right)^2} E\left[\left(\sum_{i=1}^{m-j-1} \hat{N}_{i,i+j}\right)^2 \Bigm| \mathcal{G}_j^{(m)}\right]$$

$$\overset{E[\hat{N}_{i,i+j} \mid \mathcal{G}_j^{(m)}] \overset{(5.24),(5.10)}{=} 0,\ (\cdot)^2}{=} \sum_{i=1}^{m-j-1} \frac{Var[\hat{N}_{i,i+j} \mid \mathcal{G}_j^{(m)}]}{\left(\sum_{i=1}^{m-j-1} N_{i,i+j}\right)^2} + \sum_{\substack{1 \leq i,k \leq m-j-1,\\ i \neq k}} \frac{Cov[\hat{N}_{i,i+j}, \hat{N}_{k,k+j} \mid \mathcal{G}_j^{(m)}]}{\left(\sum_{i=1}^{m-j-1} N_{i,i+j}\right)^2} \qquad (5.26)$$

$$\xrightarrow[(5.24)]{\text{(iii) of Def. 5.22},} Var[\hat{N}_{i,i+j} \mid \mathcal{G}_j^{(m)}] = Var[\hat{N}_{i,i+j} \mid \underbrace{N_{i,i}, \dots, N_{i,i+j}}_{\overset{def}{=} \mathcal{F}_{i,i+j}}]$$

and

$$Cov[\hat{N}_{i,i+j}, \hat{N}_{k,k+j} \mid \mathcal{G}_j^{(m)}] = E[\hat{N}_{i,i+j} \cdot \hat{N}_{k,k+j} \mid N_{i,i}, \dots, N_{i,i+j}, N_{k,k}, \dots, N_{k,k+j}]$$

indep. of $N_{l,l+r}$, $r \geq 0$, $l \neq i,k$

(60) On the other hand for $i<k$:

$E[\hat{N}_{i,i+j}\cdot\hat{N}_{k,k+j}\,|\,N_{i,i},\dots,N_{i,i+j},N_{k,k},\dots,N_{k,k+j}]$

$\overset{\text{tower property of cond. expect.}}{=}$

$E[E[\hat{N}_{i,i+j}\hat{N}_{k,k+j}\,|\,N_{i,i},\dots,N_{i,i+j},N_{k,k},\dots,N_{k,k+j+1}]\,|\,N_{i,i},\dots,N_{i,i+j},$
$N_{k,k},\dots,N_{k,k+j}] \overset{(5.23)}{=}$

$E[\hat{N}_{k,k+j}\underbrace{E[\hat{N}_{i,i+j}\,|\,N_{i,i},\dots,N_{i,i+j},N_{k,k},\dots,N_{k,k+j+1}]}_{\overset{(5.24)}{=}E[\hat{N}_{i,i+j}|N_{i,i},\dots,N_{i,i+j}]\overset{\text{(iii) of Def 5.22}}{=}0}\,|\,N_{i,i},\dots,N_{k,k+j}]$

cindep.

$=0$

$\overset{(5.26)}{\Longrightarrow}$ $$\mathrm{Var}[\hat{f}_j^{(m)}\,|\,\mathcal{G}_j^{(m)}]=\sum_{i=1}^{m-j-1}\frac{\mathrm{Var}[\hat{N}_{i,i+j}\,|\,N_{i,i},\dots,N_{i,i+j}]}{\left(\sum_{i=1}^{m-j-1}N_{i,i+j}\right)^2}\qquad(5.27)$$

(5.27) $\longrightarrow$ problem: $\mathrm{Var}[\hat{N}_{i,i+j}|N_{i,i},\dots,N_{i,i+j}]$ difficult or impossible to compute, in general

$\longrightarrow$ possible solution: Using a similar reasoning as in (5.8) and (5.9) for (5.10), we may require the following additional condition in Def. 5.22:

(v) For $\mathrm{Var}[N_{1,1+j}]<\infty$, $j\geq 0$ there ex. for all $j\geq 0$ a real number σ_j^2 s.t.

$$\mathrm{Var}[N_{i,i+j+1}\,|\,N_{i,i},\dots,N_{i,i+j}]=\sigma_j^2 N_{i,i+j}\qquad(5.28)$$

$\longrightarrow$ Lemma 5.26 ($\mathrm{Var}[\hat{f}_j^{(m)}]$)

Assume Mack's model with the extra condition (5.28). Then

$$\mathrm{Var}[\hat{f}_j^{(m)}]=\sigma_j^2 E\left[\frac{1}{\sum_{i=1}^{m-j-1}N_{i,i+j}}\right]\qquad(5.29)$$

Proof: We see that

$\mathrm{Var}[\hat{N}_{i,i+j}|N_{i,i},\dots,N_{i,i+j}]=\mathrm{Var}[N_{i,i+j+1}|N_{i,i},\dots,N_{i,i+j}]$
$\overset{(5.28)}{=}\sigma_j^2 N_{i,i+j}$

$\overset{(5.27)}{\Longrightarrow}$ $$\mathrm{Var}[\hat{f}_j^{(m)}\,|\,\mathcal{G}_j^{(m)}]=\frac{\sigma_j^2}{\sum_{i=1}^{m-j-1}N_{i,i+j}}\qquad(+)$$

On the other hand,

(61) $$\mathrm{Var}[\hat f_j^{(m)}] \overset{\text{excercise}}{=} E[\mathrm{Var}[\hat f_j^{(m)} \mid \mathcal{G}_j^{(m)}]] + \overbrace{\mathrm{Var}[\underbrace{E[\hat f_j^{(m)} \mid \mathcal{G}_j^{(m)}]}_{\overset{L.5.25(i)}{=} f_j}]}^{=0}$$

$$= E[\mathrm{Var}[\hat f_j^{(m)} \mid \mathcal{G}_j^{(m)}]] \overset{(+)}{\Longrightarrow} \text{proof.}$$

<u>Rem. 5.27</u> $\xrightarrow[E[\cdot]]{(5.28)}$ $E[(N_{1,1+j+1} - f_j \cdot N_{1,1+j})^2] = \sigma_j^2 \, E[N_{1,1+j}]$.

Then replacement of the expectations by their sample versions and f_j by $\hat f_j^{(m)}$ gives the estimator $[\hat\sigma_j^{(m)}]^2$ of σ_j^2:

$$[\hat\sigma_j^{(m)}]^2 = \frac{\sum_{i=1}^{m-j-1} (N_{i,1+j+1} - \hat f_j^{(m)} N_{i,1+j})^2}{\sum_{i=1}^{m-j-1} N_{i,1+j}} \qquad (5.30)$$

<u>1-step ahead prediction in Mack's model</u>

<u>Recall</u> from (5.10) that

$$E[N_{i,1+j+1} \mid N_{i,1}, \dots, N_{i,1+j}] = f_j N_{i,1+j}$$

$\xrightarrow{j=m-i}$ $E[N_{i,m+1} \mid N_{i,1}, \dots, N_{i,m}] = f_{m-i} N_{i,m}$

$\longrightarrow$ $f_{m-i} N_{i,m}$ is the best predictor of $N_{i,m+1}$ 1 year ahead (if m is the present year) in the mean square sense, that is $f_{m-i} N_{i,m}$ minimizes

distance between $N_{i,m+1}$ and Y

$$E[(N_{i,m+1} - Y)^2]$$

w.r.t. Y with $E[Y^2] < \infty$, which are functions of $N_{i,1}, \dots, N_{i,m}$

$\longrightarrow$ $\widehat{err}_N^{(m)} := E[(N_{i,m+1} - f_{m-i} N_{i,m})^2]$ minimal distance w.r.t. those Y

$\longrightarrow$ $\widehat{err}_N^{(m)}$ <u>mean square prediction error</u> of $f_{m-i} N_{i,m}$

Since $\hat f_j^{(m)} \approx f_j$ (for large $m-j$), we can approximate $f_{m-i} N_{i,m}$ by

$$\hat N_{i,m+1} := \hat f_{m-i}^{(m)} N_{i,m} \qquad (5.31)$$

(<u>1-step ahead predictor</u>)

(62) $\longrightarrow$ What is the mean square prediction error of $\hat{N}_{i,m+1}$ for 1-step ahead, that is what is

$$err_N^{(m)} := E[(\hat{N}_{i,m+1} - N_{i,m+1})^2] \quad ?$$

$\longrightarrow$ Lemma 5.28 ($err_N^{(m)}$ for 1-step ahead)
Consider Mack's model with the additional assumption (V) in (5.28). Then for $2 \le i \le m$:

$$err_N^{(m)} = \underbrace{\delta^2_{m-i} E[N_{1,1+m-i}]}_{\widetilde{err}_N^{(m)}} + Var[\hat{f}^{(m)}_{m-i}]\, E[(N_{1,1+m-i})^2]$$

$$= \delta^2_{m-i}\Big\{ E[N_{1,1+m-i}] + E\Big[\Big(\sum_{k=1}^{i-1} N_{k,k+m-i}\Big)^{-1}\Big] \cdot E[(N_{1,1+m-i})^2]\Big\}.$$

Proof: $err_N^{(m)} \overset{def}{=} E[(N_{i,m+1} - \hat{f}^{(m)}_{m-i} N_{i,m})^2] =$

$$= E[(N_{i,m+1} - f_{m-i}N_{i,m} + f_{m-i}N_{i,m} - \hat{f}^{(m)}_{m-i} N_{i,m})^2]$$

$$\overset{(\cdot)^2}{=} \underbrace{E[(N_{i,m+1} - f_{m-i}N_{i,m})^2]}_{=:D_0} + \underbrace{E[(f_{m-i} - \hat{f}^{(m)}_{m-i})^2 \cdot (N_{i,m})^2]}_{=:D_1}$$

$$+ \underbrace{2E[(N_{i,m+1} - f_{m-i}N_{i,m}) \cdot N_{i,m}(f_{m-i} - \hat{f}^{(m)}_{m-i})]}_{=:D_2}$$

By definition $\hat{f}^{(m)}_{m-i}$ is a (deterministic) function of $N_{k,k+(m-i)+1}$ and $N_{k,k+(m-i)}$, $k \le i-1$ $\overset{\text{(iii) in Def. 5.22}}{\Longrightarrow}$
$\hat{f}^{(m)}_{m-i}$ indep. of $(N_{i,m}, N_{i,m+1})$

f_j constant $\Longrightarrow$ $D_2 = 2E[(N_{i,m+1} - f_{m-i}N_{i,m})N_{i,m}] \underbrace{E[f_{m-i} - \hat{f}^{(m)}_{m-i}]}_{\overset{L.5.25(i)}{=}0} = 0$ and

$$D_1 = E[(\hat{f}^{(m)}_{m-i} - \underbrace{f_{m-i}}_{=E[\hat{f}^{(m)}_{m-i}]})^2] \cdot \underbrace{E[(N_{i,m})^2]}_{=E[(N_{1,1+m-i})^2]} = Var[\hat{f}^{(m)}_{m-i}]\, E[(N_{1,1+m-i})^2]$$

On the other hand,

$$D_0 = E[Var[N_{i,m+1} | N_{i,i}, \dots, N_{i,m}]] \overset{(5.28)}{=} \delta^2_{m-i} \underbrace{E[N_{i,m}]}_{=E[N_{1,1+m-i}]}$$

$\overset{(5.29)}{\Longrightarrow}$ proof.

Rem. 5.29: (i) In the representation of $err_N^{(m)}$, δ^2_{m-i} can be approximated by $[\hat{\delta}^{(m)}_{m-i}]^2$ in (5.30) and $E[(N_{1,1+m-i})^2]$, $E[N_{1,1+m-i}]$ by their sample versions, if i and m are large.
(ii) Precise bounds for $Var[\hat{f}^{(m)}_{m-i}]$ are difficult to derive. However, since $N_{1,1+j} \ge 1$ with prob. 1, it follows

63 from (5.29) that

$$\mathrm{Var}[\hat{f}_{m-i}^{(m)}] \le \sigma_{m-i}^2/(i-1)$$

$\longrightarrow$ In order to get $err_N^{(m)}$ small, i should be large

K-step ahead prediction in Mack's model

We are now interested in the prediction of future claim numbers $N_{i,m+k}$ based on $N_{i,1}, \dots, N_{i,m}$ (m present year)

$\longrightarrow$ $E[N_{i,m+k} \mid N_{i,1}, \dots, N_{i,m}]$ best predictor of $N_{i,m+k}$ k years ahead in the mean square sense

(5.10) $E[E[X \mid \mathcal{B}] \mid \mathcal{A}] = E[X \mid \mathcal{A}]$, $\mathcal{A} \subseteq \mathcal{B}$

$$E[N_{i,m+k} \mid N_{i,1}, \dots, N_{i,m}] =$$

$$= E[\underbrace{E[N_{i,m+k} \mid N_{i,1}, \dots, N_{i,m+k-1}]}_{\overset{(5.10)}{=} f_{m-i+k-1} N_{i,m+k-1}} \mid N_{i,1}, \dots, N_{i,m}]$$

$$= f_{m-i+k-1} E[N_{i,m+k-1} \mid N_{i,1}, \dots, N_{i,m}] = \dots$$

$$= f_{m-i+k-1} f_{m-i+k-2} \cdots f_{m-i} N_{i,m} \qquad (5.32)$$

Since $\hat{f}_j^{(m)} \approx f_j$ (for large $m-j$), we can approximate the right hand side of (5.32) by

$$\hat{N}_{i,m+k} := \hat{f}_{m-i+k-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)} N_{i,m} \qquad (5.33)$$

(k-step ahead predictor)

$\longrightarrow$ What is the mean square prediction error of $\hat{N}_{i,m+k}$, i.e. $err_N^{(m)} := E[(\hat{N}_{i,m+k} - N_{i,m+k})^2]$?

$\longrightarrow$ Lemma 5.30 ($err_N^{(m)}$ for k-steps ahead)
Consider Mack's model with the additional condition (5.28). Then

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error estimate w.r.t. (5.32)

$$\mathrm{err}_N^{(m)} = E[(N_{i,m+k} - E[N_{i,m+k} | N_{i,i}, \dots, N_{i,m}])^2] =: \mathcal{J}_1$$
$$+ E[(E[N_{i,m+k} | N_{i,i}, \dots, N_{i,m}] - \hat{N}_{i,m+k})^2] =: \mathcal{J}_2$$

Moreover,

$$\mathcal{J}_1 = E[N_{1,1+m-i}] \cdot \sum_{\ell=m-i}^{m+k-i-1} (f_{m-i+k-1} \cdots f_{\ell+1})^2 \sigma_\ell^2 f_{\ell-1} \cdots f_{m-i} \qquad (5.34)$$

and

$$E[(N_{1,1+m-i})^2] \sum_{\ell=m-i}^{m+k-i-1} (f_{m-i+k-1} \cdots f_{\ell+1})^2 \sigma_\ell^2 f_{\ell-1} \cdots f_{m-i} \cdot E[(\sum_{r=1}^{i-1} N_{r,r+m-i})^{-1}]$$
$$\leq \mathcal{J}_2 \leq E[(N_{1,1+m-i})^2] \cdot \sum_{\ell=m-i}^{m+k-i-1} (f_{m-i+k-1} \cdots f_{\ell+1})^2 a_\ell, \qquad (5.35)$$

where

$$a_\ell := \frac{\sigma_\ell^2}{m-\ell-1} \left(\frac{\sigma_{\ell-1}^2}{m-\ell} + f_{\ell-1}^2\right) \cdots \left(\frac{\sigma_{m-i+1}^2}{i-2} + f_{m-i+1}^2\right) \cdot$$
$$\cdot \left(\sigma_{m-i}^2 E[(\sum_{r=1}^{i-1} N_{r,r+m-i})^{-1}] + f_{m-i}^2\right).$$