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2. Some prerequisites

Denote by T the future life time of a person or the first arrival time of a claim

$\longrightarrow$ outcome of T uncertain

$\longrightarrow$ model for T:

T random variable (r.v.) on a probability space $(\Omega, \mathcal{F}, P)$

1. Ω:

Def. 2.1 (Sample space)

A set $\Omega \neq \emptyset$ representing the collection of all possible outcomes of a random experiment is called sample space.

Ex.: Throw a die $\longrightarrow \Omega = \{1, 2, \dots, 6\}$

2. $\mathcal{F}$:

Def. 2.2 (σ-algebra on Ω)

A family $\mathcal{F}$ of subsets of Ω is called σ-algebra (on Ω) if

(i) $\emptyset \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ ($A^c \overset{\text{def.}}{=} \Omega \setminus A$)

(iii) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i \geq 1} A_i \in \mathcal{F}$

Interpretation: $\mathcal{F}$ is e.g. a model for the entirety of market information up to time t.

The elements $A \in \mathcal{F}$ are called events.

Ex.: Let S_t be the price of a stock at time t.

$\longrightarrow A := "S_t \geq 100\$"$ market event

Ex.: $A = \emptyset$ impossible event as e.g. $A = "S_t = \infty" \in \mathcal{F}$ (Def 2.2 (i)) $\longrightarrow A^c = "0 \leq S_t < \infty" \in \mathcal{F}$ (Def. 2.2 (ii))

Ex.: Denote by $S_t^{(i)}$, $i \in \mathbb{N}$ the price of the i-th stock at time t

Consider the events

$A_i = "S_t^{(i)} \geq 10\$"$

⑩ $\Rightarrow$ $A :=$ "at least one stock has a price ≥ 10 \$ at time t"
$= \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$ (Def 2.2 (iii))

Ex.: $\Omega = \{1, 2, \ldots, 6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ class of all subsets of Ω
$A = \{1, 2, 3\} \in \mathcal{F}$ event that one throws 1, 2 or 3

Ex.: (Borel σ-algebra $\mathcal{B}(\mathbb{R})$)
$\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, where
$\mathcal{B}(\mathbb{R}) \overset{def}{=}$ smallest σ-algebra, which contains the intervals $[a, b)$, $a \leq b$
$=$ intersection of all σ-algebras $\mathcal{F}_i$ containing the intervals $[a, b)$, $a \leq b$

3. P:

Def. 2.3 (Probability measure)
A function
$$P : \mathcal{F} \longrightarrow [0, 1]$$
is called probability measure on $(\Omega, \mathcal{F})$, if
(i) $P(\emptyset) = 0$, $P(\Omega) = 1$
(ii) $A_1, \ldots, A_n, \ldots \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$, $i \neq j$ (disjoint)
$\Rightarrow P(\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} P(A_i)$ (σ-additivity)

Interpretation: $P(A) =$ "the prob. that the event A occurs"

Useful properties:
(i) $A \subseteq B \Rightarrow P(A) \leq P(B)$
(e.g. $A =$ "$S_t \geq 100$ \$" $\subseteq B =$ "$S_t \geq 90$ \$")
(ii) $P(\bigcup_{i \geq 1} A_i) \leq \sum_{i \geq 1} P(A_i)$ for not necessarily disjoint events
(e.g. $A_i =$ "$S_t^{(i)} \geq 10$ \$" $\Rightarrow A = \bigcup_{i \geq 1} A_i =$ "at least one stock ≥ 10 \$ at time t")

(11) (iii) $\lim_{i\to\infty} P(A_i) = P(A)$, if $A_1 \subset A_2 \subset \dots$ and

$$A = \bigcup_{i \geq 1} A_i$$

(e.g. $A_i = $ "$S_t \geq \frac{100}{i}$ \$", $i \in \mathbb{N}$)

Examples of prob. meas.

a) $\Omega = \{1, 2, \dots, 6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ class of all subsets of Ω

$P(A) \overset{\text{def}}{=}$ (number of elements in A)/6

e.g. $P(\{1,2,3\}) = \frac{1}{2}$ $\longrightarrow$ prob. that one throws 1, 2 or 3

b) Dirac measure δ_ω

$$\delta_\omega(A) \overset{\text{def}}{=} \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

c) Lebesgue (-Borel) measure on $[0,1]$

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}(\mathbb{R}) \cap [0,1] = \{A \cap [0,1] : A \in \mathcal{B}(\mathbb{R})\}$

One shows that there exists a unique prob. meas.

$$\lambda : \mathcal{F} \longrightarrow [0,1] \text{ s.t.}$$

$$\lambda([a,b)) = b - a$$

$\uparrow$ length of the interval $[a,b)$

for all $0 \leq a \leq b \leq 1$.

Ex.: S uniformly distributed on $[0,1]$

$\Longleftrightarrow$ $P(a \leq S \leq b) = b - a = \lambda([a,b))$

$\Longrightarrow$ distr of S = Lebesgue meas. on $[0,1]$

Def 2.4 (Random variable)

A function

$$X : \Omega \longrightarrow \mathbb{R}$$

is called random variable (r.v.) on $(\Omega, \mathcal{F}, P)$, if the set

$$A := \{\omega \in \Omega : X(\omega) < t\}$$

is an event for all $t \in \mathbb{R}$, i.e.

$A \in \mathcal{F}$ (collection of all events)

for all t

Ex. $\mathcal{F}$ entirety of market information up to time T

$\longrightarrow$ natural to assume that

$$\{S_T < t\} \leftarrow \text{stock price at time } T$$

is a (possible) market event for all t (\$), i.e.

$$\{S_T < t\} \in \mathcal{F} \text{ for all } t$$

$\Longrightarrow$ S_T random variable

<u>Def. 2.5</u> (Random vector)

A function

$$X : \Omega \longrightarrow \mathbb{R}^n$$

with

$$X(\omega) = (X_1(\omega), \dots, X_n(\omega))$$

is called <u>random vector</u>, if $X_1, \dots, X_n$ are r.v.'s

<u>Def. 2.6</u> (Stochastic process)

Let $\mathcal{J} \neq \emptyset$ be an index set (e.g. $\mathcal{J} = [0,1]$)

A collection

$$\{X_i\}_{i \in \mathcal{J}}$$

of random variables is called <u>stochastic process</u>.

$\mathcal{J}$ is the <u>parameter space</u>

<u>Ex.</u>: $\mathcal{J} = [0, \infty)$, S_t price of a stock at time t

$\longrightarrow$ $\{S_t\}_{0 \leq t < \infty}$ stoch. proc.

<u>Def. 2.7</u> (P-null sets)

An event

$$A \in \mathcal{F}$$

s.t.

$$P(A) = 0$$

is called <u>P-null set</u>

<u>Ex.</u>: $A = \{S_t = \infty\}$ should be a P-null set, i.e. $P(A) = 0$

<u>Def. 2.8</u> (P-almost everywhere property)

We say a property Π holds <u>P-almost everywhere</u> (P-a.e.)

if there ex. a null set $N \in \mathcal{F}$ s.t.

Π holds for all $\omega \in N^c = \Omega \setminus N$

(13) Ex.: X_i: i-th fire loss claim, $i=1,2$

$\Pi = $ " $X_1 \le X_2$ "

$\longrightarrow$ Π holds P-a.e. $\Longleftrightarrow$

$$X_1(\omega) \le X_2(\omega)$$

for all $\omega \in N^c = \Omega \setminus N$ for some null set $N \in \mathcal{F}$.

Def. 2.9 (Integral w.r.t. P)

Let $X \ge 0$ be a non-negative r.v. Then

$$E[X] \stackrel{\text{def.}}{=} \int_\Omega X(\omega)\, P(d\omega) \quad \leftarrow \text{notation}$$

$$\stackrel{\text{def.}}{=} \lim_{n\to\infty} \sum_{i=1}^{n\cdot 2^n} i \cdot 2^{-n} P(A_{i,n}),$$

where
$$A_{i,n} := \begin{cases} \{\omega \in \Omega : (i+1)2^{-n} > X(\omega) \ge i\cdot 2^{-n}\}, & i=0,\dots,n2^n-1 \\ \{\omega\in\Omega : X(\omega) \ge n\}, & i = n\cdot 2^n \end{cases}$$

For general X we def.

$$E[X] := \int_\Omega X(\omega)\, P(d\omega) :=$$

$$E[X^+] - E[X^-] \quad (x^+ = \max(x,0),\ x^- = \max(-x,0)),$$

provided $E[X^+], E[X^-] < \infty$ (*)

If (*) holds, X is called (P-) integrable.

Ex.: $\Omega = [0,1]$, $P = \lambda$ (uniform distribution)

$$X(\omega) = \omega^2$$

$$\Rightarrow \quad E[X] = \int_\Omega X(\omega)\, P(d\omega) = \int_{[0,1]} \omega^2 \lambda(d\omega)$$

$$= \int_0^1 \omega^2 d\omega = \frac{1}{3}\omega^3 \Big|_{\omega=0}^{1} = \frac{1}{3}$$

Properties:

(i) $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$, $\alpha,\beta \in \mathbb{R}$ (linearity)

(ii) If $X \le Y$ P-a.e. then

$$E[X] \le E[Y]$$

(14) Ex.: $X \in \mathbb{N}_0$ r.v.

$$\Rightarrow X = \sum_{n \geq 0} n \cdot \underbrace{\mathbb{1}_{\{X=n\}}}_{= Y_n \text{ r.v.'s}} = \sum_{n\geq 0} n \cdot Y_n$$

$$\mathbb{1}_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0 & \text{else} \end{cases}$$

Using (i) we obtain that

$$E[X] = E[\sum_{n\geq 0} n \cdot Y_n] = \sum_{n \geq 0} n \underbrace{E[Y_n]}_{= P(\{X=n\})}$$

$$= \sum_{n \geq 0} n \cdot P(\{X=n\})$$

Fundamental concept of prob. th $\longrightarrow$

Conditional expectation $E[X|\mathcal{A}]$ of a r.v. X w.r.t. a σ-algebra $\mathcal{A}$ ($\subset \mathcal{F}$)

Heuristically:

$$E[X|\mathcal{A}] = \text{"} E[X|A], A \in \mathcal{A} \text{"},$$

where $E[X|A] \overset{\text{def.}}{=} \frac{1}{\underset{0 \neq}{P(A)}} E[\mathbb{1}_A \cdot X]$ cond. expectation of X given the event A

Interpretation: $\mathcal{A}$ entirety of market inf. up to time $t<T$ maturity

$\longrightarrow$ $\mathcal{A} \subseteq \mathcal{F}$ market inf. up to time T.

S_T stock price at time T

$\longrightarrow$ $E[S_T|\mathcal{A}]$ expected terminal stock price given the entirety of market inf. up to time $t<T$.

Or more mathematically:

Def 2.10 (Cond. expectation w.r.t $\mathcal{A}$) $\leftarrow$ in general difficult to calculate

Let X be a r.v. s.t. $E[|X|]<\infty$

Then the expected value of X given $\mathcal{A}$ is the

(15) <u>unique</u> r.v. Y s.t.

$$E[\mathbb{1}_A \cdot X] = E[\mathbb{1}_A \cdot Y]$$

for all $A \in \mathcal{A}$

and $\{Y \leq q\} \in \mathcal{A}$ for all $q \in \mathbb{R}$.

The r.v. Y is denoted by $E[X|\mathcal{A}]$

→ <u>Properties of $E[X|\mathcal{A}]$</u> !

(i) $E[\alpha X + \beta Y | \mathcal{A}] = \alpha E[X|\mathcal{A}] + \beta E[Y|\mathcal{A}]$ (linearity)

(ii) $E[E[X|\mathcal{A}]] = E[X]$

(iii) $E[X|\mathcal{A}] = X$, if X r.v. on $(\Omega, \mathcal{A}, P)$

(iv) $E[X|\mathcal{A}] = E[X]$, if X is <u>independent</u> of $\mathcal{A}$, i.e.
$P(\{X \leq t\} \cap A) = P(X \leq t) \cdot P(A)$ for all $t \in \mathbb{R}$, $A \in \mathcal{A}$

(v) $E[X|\mathcal{A}] = E[E[X|\mathcal{B}]|\mathcal{A}]$ if $\mathcal{A} \subseteq \mathcal{B}$ σ-algebra