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3. Ruin theory

→ objective: study of ruin probabilities, i.e. of probabilities for ruin in an insurance portfolio in the case of both small claims and large claims

For this purpose, we have to introduce some basic notions of ruin theory:

3.1 Risk process, Ruin probability, Net Profit Condition

Def. 3.1.1 (Renewal model for $S(t)$)

The total claim amount $S(t)$ in the renewal model is defined by

$$S(t) = \sum_{i=1}^{N(t)} X_i, \qquad (3.1.1)$$

where

(i) $X_i, i \geq 1$ are i.i.d non-negative r.v.'s. Here X_i models the i-th claim size (e.g. wind storm loss).

(ii) $N(t), t \geq 0$ is the renewal (counting) process, which counts the number of claims arriving by time t and which is defined as

$$N(t) = \underbrace{\#}_{\text{number of}}\{i \geq 1 : T_i \leq t\}, \quad t \geq 0$$

where

$$T_0 := 0, \; T_n = W_1 + \dots + W_n$$

for an i.i.d sequence $W_i, i \geq 1$ of (P-a.e.) positive r.v.'s.

The r.v. T_i is called the i-th arrival time (of claim X_i), whereas $W_i = T_i - T_{i-1}$ is referred to as the i-th inter-arrival time

and

(iii) $W_i, i \geq 1$ independent of $X_i, i \geq 1$

Recall: $Y_1, \dots, Y_n$ r.v.'s indep. $\Longleftrightarrow$ $P(Y_1 \leq x_1, \dots, Y_n \leq x_n) = P(Y_1 \leq x_1) \cdot \ldots \cdot P(Y_n \leq x_n)$ for all $x_1, \dots, x_n \in \mathbb{R}$.

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Rem. 3.1.2 (Poisson proc. as an example of a renewal proc.)

Consider a homogeneous Poisson process, that is a stoch. proc. $N(t), t \geq 0$ on $(\Omega, \mathcal{F}, P)$ s.t.

(i) $N(0) = 0$ P-a.e.

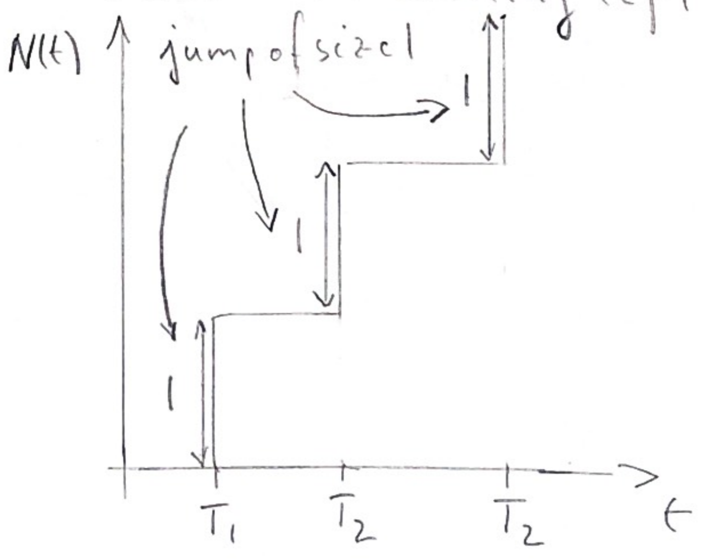
(ii) N has independent increments:
$$N(t_1) - N(t_0), \dots, N(t_n) - N(t_{n-1})$$
are indep. r.v.'s for all $0 = t_0 < t_1 < \dots < t_n$, $n \in \mathbb{N}$.

(iii) N has stationary Poisson distributed increments:
$$N(t+h) - N(t) \overset{d}{=} N(h) \quad \text{with } N(h) \sim \text{Pois}(\lambda \cdot h)$$
for all $t \geq 0$, $h > 0$ for some $\lambda > 0$ (jump intensity)

Recall:

1. X, Y r.v.'s $\Rightarrow$ $X \overset{d}{=} Y \iff P(X \leq x) = P(Y \leq x)$ for all x.
2. $M \sim \text{Pois}(\mu) \iff P(M = k) = e^{-\mu} \frac{\mu^k}{k!}$, $k \in \mathbb{N}_0$

(iv) With probability 1 the sample paths of $N(t, \omega)$, $t \geq 0$ are cadlag, that is $\exists\, \hat{\Omega} \in \mathcal{F}$ with $P(\hat{\Omega}) = 1$ s.t. for all $\omega \in \hat{\Omega}$:
$$(t \mapsto N(t, \omega))$$
right-continuous with existing left limits



(18) One shows that

(i) If N is a hom. Poisson proc. with intensity λ and arrival times $0 \le T_1 \le T_2 \le \dots$ then
N is a renewal proc. with i.i.d $W_i \sim Exp(\lambda)$, $i \ge 1$, that is

$$P(W_i \le x) = \int_0^x f(y)\,dy \qquad (3.1.2)$$

with probability density

$$f(y) = \lambda e^{-\lambda y}, \quad y \ge 0$$

(ii) Let N be a renewal process with i.i.d $W_i \sim Exp(\lambda)$.
Then N is a hom. Poisson proc. with intensity $\lambda > 0$.

<u>Ex. 3.1.3</u> (<u>Cramér-Lundberg-model</u> as an example of a renewal model)

If $N(t)$, $t \ge 0$ is a hom. Poiss. proc. (Def. 3.1.1), then $S(t)$, $t \ge 0$ is the total claim amount in the <u>Cramér-Lundberg-model</u>.

<u>Def. 3.1.4</u> (Risk process)

The <u>risk process</u> (or surplus) of the portfolio is defined as

$$U(t) = u + p(t) - S(t), \quad t \ge 0,$$

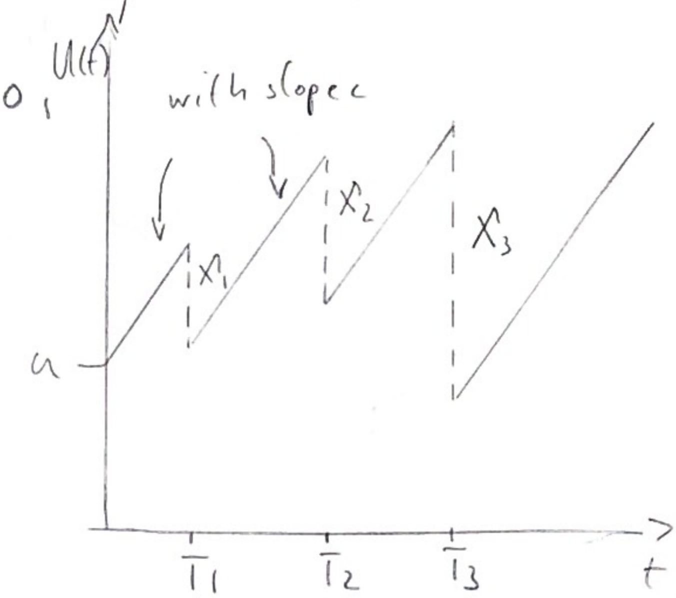
where

$u > 0$ <u>initial capital</u> (sufficiently large to avoid bankruptcy) ! e.g. reinforced by supervisory authorities!

$p(t)$ <u>premium income</u>

<u>Assumption</u>: $p(t) = c \cdot t$, $t \ge 0$,

where $c > 0$ <u>premium rate</u>



(19) <u>Def. 3.1.5</u> (Ruin, ruin time, ruin probability)

(i) The event
$$\text{Ruin} := \{U(t) < 0 \text{ for some } t \geq 0\}$$
is called <u>ruin</u>

$\longrightarrow$ event that U ever falls below 0

(ii) The r.v.
$$T := \inf\{t \geq 0 : U(t) < 0\} \in [0, \infty] \quad (\inf \emptyset \overset{\text{def.}}{=} \infty)$$
is called <u>ruin time</u>

$\longrightarrow$ time when ruin occurs

(iii) The <u>probability of ruin</u> is def. as
$$\psi(u) := \underbrace{P(\text{Ruin} \mid U(0) = u)}_{= P(\text{Ruin})} = P(T < \infty)$$

Note that
$$\text{Ruin} = \bigcup_{t \geq 0} \{U(t) < 0\} = \{\inf_{t \geq 0} U(t) < 0\}$$
$$= \{T < \infty\}$$

$\longrightarrow$ <u>main objective</u> of insurance business is to keep $\psi(u)$ small for sufficiently large initial capital u

worst case $\longrightarrow$

<u>Th. 3.1.6</u> (Ruin with prob. 1)

Let $E[W_1], E[X_1] < \infty$ and
$$E[X_1] - c \cdot E[W_1] \geq 0 \quad (3.1.3)$$

Then
$$P(\text{Ruin} \mid U(0) = u) = 1$$
for all $u > 0$.

<u>Ex. 3.1.7</u> : <u>Recall</u> the premium $p(t)$ based on the equivalence principle, that is
$$p(t) = p_{Net}(t) = E[S(t)]$$

$N(t)$ indep. of the i.i.d X_i, $i \geq 1$ (Def. 3.1.1 (iii))

$\Rightarrow$ $E[S(t)] = E[N(t)] \cdot E[X_1]$ (see exerc.)

(20) Suppose $N(t)$ hom. Poisson proc. with intensity $\lambda > 0$,

then $E[S(t)] = \underbrace{\lambda \cdot t}_{= E[N(t)]} E[X_1] = \frac{E[X_1]}{E[W_1]} \cdot t$,

since $E[W_1] = \frac{1}{\lambda}$ (see 3.1.2)

$\int_0^\infty y \cdot \lambda e^{-\lambda y} dy$

$\Rightarrow c = \frac{E[X_1]}{E[W_1]} \Rightarrow (3.1.3) = 0 \Rightarrow$

P_{net} leads to ruin for all $u > 0$

$\Rightarrow$ P_{net} unwise calculation principle

<u>Proof of Th. 3.1.6</u> : idea: Strong law of large numbers (SLNN)

Rewrite

$$\text{Ruin} = \{\inf_{t>0} U(t) < 0\} \overset{U(t)\uparrow \text{ on } [T_n, T_{n+1})}{=} \{\inf_{n \geq 0} U(T_n) < 0\}$$

$$\overset{\text{def. of } U}{=} \{\inf_{n\geq 1} [u + P(T_n) - S(T_n)] < 0\} \overset{S(t) = \sum_{i \geq 1} X_i \mathbb{1}_{[0,t]}(T_i)}{=}$$

$$= \{\inf_{n\geq 1} [u + \underbrace{(c \cdot T_n - \sum_{i=1}^n X_i)}_{= -(Z_1 + \ldots + Z_n) =: -S_n}] < 0\}$$

with $Z_i := X_i - c \cdot W_i$ (since $T_n = \sum_{i=1}^n W_i$)

$$\Rightarrow \psi(u) = P(\inf_{n\geq 1}(-S_n) < -u) = P(\sup_{n \geq 1} S_n > u)$$

$$\overset{\text{SLLN}}{\underset{Z_i,\, i\geq 1 \text{ i.i.d}}{\Longrightarrow}} \frac{S_n}{n} = \frac{Z_1 + \ldots + Z_n}{n} \xrightarrow[n\to\infty]{} E[Z_1] = E[X_1] - cE[W_1] \geq 0$$

(3.1.4)

<u>1. case</u> : $E[Z_1] > 0 \overset{(3.1.4)}{\Longrightarrow} S_n \xrightarrow[n\to\infty]{} \infty$ with prob. 1

$\Rightarrow \psi(u) = P(\sup_{n \geq 1} S_n > u) = 1 \quad \forall u$

<u>2. case</u> : $E[Z_1] = 0$. See e.g. Spitzer (1976): Principles of Random Walk.

$\Rightarrow$ proof.

(21) What happens if $E[X_1] - c \cdot E[W_1] < 0$?

$\longrightarrow$ ruin may still occur, but not necessarily with prob. 1

$\longrightarrow$ Def. 3.1.7 (Net profit condition)

$S(t)$ satisfies the net profit condition (NPC) if

$$E[X_1] - c \cdot E[W_1] < 0 \qquad (3.1.5)$$

Under NPC one can at least obtain an upper bound for $\psi(u)$ for small claims!

3.2. Lundberg's inequality

Assumption: $S(t)$ is described by the renewal model (Def. 3.1.1) s.t. NPC holds (Def. 3.1.7) as well as the small claim condition is satisfied, that is

$$m_{X_1}(h) := E[e^{h \cdot X_1}] < \infty, \quad h \in (-h_0, h_0) \text{ for some } h_0 > 0$$

(moment generating function) (3.2.1)

$\longrightarrow$ Markov's inequality for $h \in (0, h_0)$:

$$P(X_1 > x) \le e^{-h \cdot x} m_{X_1}(h) \quad \text{for all } x > 0$$

$\Longrightarrow$ tail distribution $P(X_1 > x)$ decays to 0 exponentially fast

$\longrightarrow$ not realistic property in many applications, where claims are rather heavy-tailed.

| Recall: Markov's inequality
| $g \geq 0$ increasing, $X \geq 0$ r.v., $x \geq 0$ with $g(x) > 0$
| $\Longrightarrow \quad P(X \geq x) \le \frac{1}{g(x)} \cdot E[g(X)]$
| Here: $g(x) = e^{h \cdot x}$, $X = X_1$

Def. 3.2.1 (Adjustment or Lundberg coefficient)

Suppose that $m_{Z_1}(h) < \infty$ on $(-h_0, h_0)$, $h_0 > 0$ for $Z_1 := X_1 - cW_1$ and that there ex. a unique $r > 0$ s.t.

$$m_{Z_1}(r) = E[e^{r(X_1 - cW_1)}] = 1 \qquad (3.2.2)$$

Then r is called the adjustment or Lundberg coefficient.