22 Rem. 3.2.2: Sufficient condition for (3.2.2)

$\exists\, 0 < h_1 < \infty : m_{Z_1}(h) < \infty$ for $h < h_1 \wedge \lim_{h \uparrow h_1} m_{Z_1}(h) = \infty$

Th. 3.2.3 (Lundberg inequality)

Consider the renewal model with NPC (Def. 3.1.7). Suppose that the adjustment coefficient r in Def. 3.2.1 exists. Then for all $u > 0$:

$$\psi(u) \leq e^{-r \cdot u} \qquad (3.2.3)$$

Rem. 3.2.4: The inequality (3.2.3) implies that large initial capital u (depending on the magnitude of the adjustment coeff.) leads to a small ruin probability.

$\longrightarrow$ Roughly speaking: Principally, there will be no danger of ruin, if the small claim condition holds and if there is large initial capital

Proof of Th. 3.2.3: Proof by induction $\quad S_k \overset{def}{=} Z_1 + \dots + Z_k$

Def. $\quad \psi_n(u) = P(\max_{1 \leq k \leq n} S_k > u) = P(\exists k \in \{1, \dots, n\} : S_k > u)$

We know from the proof of Th. 3.1.6 that

$$\psi(u) = P(\exists k \geq 1 : S_k > u)$$

$\overset{\text{property (iii) of prob. meas.}}{\Longrightarrow} \quad \psi_n(u) \uparrow \psi(u)$ for $n \longrightarrow \infty$

$\Longrightarrow$ sufficient to show that

$$\psi_n(u) \leq e^{-r \cdot u} \text{ for all } n \geq 1 \text{ and } u > 0 \quad (3.2.4)$$

$n = 1$: $\psi_1(u) = P(Z_1 > u) \overset{\text{Markov ineq for } g(y) = e^{ry}}{\leq}$

$$\leq e^{-ru} \underbrace{m_{Z_1}(r)}_{\overset{(3.2.2)}{=} 1} = e^{-ru}$$

Assume that (3.2.4) holds for $n = k \geq 1$. Then

$\psi_{k+1}(u) = P(\max_{1 \leq n \leq k+1} S_n > u) \quad =$

(23)

$$= \underbrace{P(Z_1 > u, \max_{1\le n\le k+1} S_n > u)}_{= P(Z_1 > u)} + P(Z_1 \le u, \max_{1\le n\le k+1} S_n > u)$$

$$= P(Z_1 > u) + P(\max_{1\le n\le k+1} \overset{= Z_1 + (S_n - Z_1)}{S_n} > u, Z_1 \le u)$$

$$= P(Z_1 > u) + P(\max_{2\le n\le k+1} (Z_1 + (S_n - Z_1)) > u, Z_1 \le u) \quad (*)$$

w.l.o.g. let $k=1$. Then

$$(*) = P(Z_1 > u) + P(Z_1 + (S_2 - Z_1) > u, Z_1 \le u)$$

$$\overset{Z_i \text{ i.i.d.}}{\underset{\text{dobbel-forventning}}{=}} \underbrace{P(Z_1 > u)}_{=: p_1} + \underbrace{E[P(x + Z_2 > u, x \le u)|_{x=Z_1}]}_{=: p_2}$$

As for p_2 we see that

$$p_2 = E[P(Z_2 > u - x, x \le u)|_{x=Z_1}]$$

$$= E[(1\!\!1_{(-\infty,u]}(x) \cdot \underbrace{P(Z_2 > u-x)}_{= P(Z_1 > u-x) \overset{\text{def.}}{=} \psi_1(u-x)})|_{x=Z_1}]$$

induction assumption $\le$ for $n=k=1$

$$\le E[(1\!\!1_{(-\infty,u]}(x)\, e^{r(u-x)})|_{x=Z_1}] = E[1\!\!1_{(-\infty,u]}(Z_1)\, e^{r(Z_1-u)}]$$

On the other hand,

$$p_1 \overset{\text{def}}{=} P(Z_1 > u) = E[1\!\!1_{(u,\infty)}(Z_1)] \le E[1\!\!1_{(u,\infty)}(Z_1) \underbrace{e^{\overset{\ge 0}{r(\underset{\ge 0}{Z_1 - u})}}}_{\ge 1}] \quad (**)$$

$$\overset{(*),(**)}{\Longrightarrow} \psi_2(u) = p_1 + p_2 = E[\underbrace{(1\!\!1_{(-\infty,u]}(Z_1) + 1\!\!1_{(u,\infty)}(Z_1))}_{=1} e^{r(Z_1-u)}]$$

$$= e^{-ru} \underbrace{E[e^{rZ_1}]}_{= m_{Z_1}(r) = 1} = e^{-ru} \Longrightarrow$$

induction for $n = k+1 = 2$ $\Longrightarrow$ proof.

3.3 Exact asymptotics of $\psi(u)$ for small claims

→ objective : bounds for $\psi(u)$ from above and below in the case of the Cramér-Lundberg model, that is in the case, when $N(t), t \geq 0$ in the renewal model is described by a hom. Poisson proc. with an intensity $\lambda > 0$.

→ one of the most important results in ruin theory

Th. 3.3.1 (Cramér's bound)
Consider the Cramér-Lundberg model (Ex. 3.1.3)
Suppose that the following conditions are satisfied:

(i) NPC in (3.1.5)

(ii) The claim size distr. F_{X_1} has a prob. density, i.e.
$$F_{X_1}(x) \overset{def.}{=} P(X_1 \leq x) = \int_{-\infty}^{x} f(y)\, dy \quad \text{for all } x$$
for a function (i.e. probability density) $f = f_{X_1} \geq 0$

(iii) $m_{X_1}(h) \overset{def.}{=} E[e^{hX_1}]$ exists on some $(-h_0, h_0)$

(iv) The adjustment coefficient r (Def. 3.2.1) exists in $(0, h_0)$.

Then there exists a $C > 0$ s.t.
$$\lim_{u \to \infty} e^{r \cdot u} \psi(u) = C.$$

Moreover, the constant C is given by
$$C = \left[\frac{r}{\rho\, E[X_1]} \int_0^{\infty} x\, e^{r \cdot x} P(X_1 > x)\, dx \right]^{-1},$$
where $\rho > 0$ is the safety loading w.r.t. a premium income defined by
$$p(t) = (1+\rho) E[S(t)] = \underbrace{\left((1+\rho) \frac{E[X_1]}{E[W_1]} \right)}_{= c \text{ premium rate}} \cdot t \qquad \text{satisfies the NPC}$$

25 Ex. 3.3.2 ($X_1 \sim Exp(\gamma)$)

If $X_1 \sim Exp(\gamma)$, $\gamma > 0$ in Th. 3.3.1, then

$$e^{Tu}\psi(u) = \frac{1}{1+\rho} \qquad (3.3.1)$$

See exercises

Proof: The proof is based on Smith's key renewal theorem and the fundamental integral equation for non-ruin probabilities (L.3.3.3).

Lemma 3.3.3 (Fundamental integral eq. for non-ruin probabilities)

Denote by $\varphi(u) := 1 - \psi(u)$ (non-ruin probability)

and $\overline{F}_X = 1 - F_X$ (tail distribution)

for r.v.'s X.

Assume the Cramér-Lundberg model for $S(t)$ s.t. NPC and $E[X_1] < \infty$ are satisfied. Further, require that F_{X_1} has a prob. density.

Then the non-ruin probability $\varphi(u)$ solves the integral eq.

$$\varphi(u) = \varphi(0) + \frac{1}{(1+\rho)E[X_1]} \int_0^u \underbrace{\overline{F}_{X_1}(y)}_{=P(X_1 > y)} \varphi(u-y)\,dy \qquad (3.3.2)$$

Proof: See T. Mikosch