3.4 Exact asymptotics of $\Psi(u)$ in the case of large claims

The small claim condition (3.2.1) may be reasonable in connection with claims in water insurance. Examples of claim size distributions satisfying (3.2.1) are

$\bar F(x)$ (i) $\bar F(x) = e^{-\lambda x}, x \geq 0, \lambda > 0$
(exponential distr.)

prob. density (ii) $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x \geq 0, \alpha, \beta > 0$ (3.4.1)
(Gamma distr.)

(iii) $\bar F(x) = e^{-c x^\tau}, x \geq 0, c > 0, \tau \geq 1$ (3.4.2)
(Weibull distr.)

$\longrightarrow$ the tail distributions decay to zero exponentially fast for $x \to \infty$

However, claim size data in fire insurance or windstorm insurance indicate that the true claim size distr. F_{X_1} does not fulfill (3.2.1), but is rather heavy-tailed, that is subexponential. Examples of such distr. are exponential transformations of r.v.'s with distr. of the type (3.2.1), that is e.g. of normally, exponentially or Gamma-distr. r.v.'s:

(i) $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma \cdot x} e^{-(\log x - \mu)^2/(2\sigma^2)}, x \geq 0, \mu \in \mathbb{R}, \sigma > 0$ (3.4.3)
(log-normal distr.)

(ii) $\bar F(x) = \left(\frac{\varkappa}{\varkappa + x}\right)^\alpha, x \geq 0, \alpha, \varkappa > 0$ (3.4.4)
(Pareto-distr.)

(iii) $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} ((\log x))^{\beta-1} \cdot x^{\alpha-1}, x \geq 0, \alpha, \beta > 0$ (3.4.5)
(log-Gamma distr.)

Other examples:

(iv) $\bar F(x) = \left(\frac{\varkappa}{\varkappa + x^\tau}\right)^\alpha, x \geq 0, \alpha, \varkappa, \tau > 0$ (3.4.6)
(Burr-distr.)

(v) $\bar F(x) = e^{-c x^\tau}, x \geq 0, c > 0, 0 < \tau < 1$ (3.4.7)
(Weibull-distr.)

$\longrightarrow$ objective: Calculation of the exact asymptotics of $\Psi(u)$ w.r.t. the Cramér-Lundberg model for large claims, that is for subexponential claim size distr.

In doing so we have to introduce the concept of subexponential distributions

For this purpose, we need some notions and auxiliary results from the theory of regularly varying functions

Def. 3.4.1 (Slowly varying functions)

A (measurable) function $L:(0,\infty)\longrightarrow(0,\infty)$ is called slowly varying (at infinity), if for all $c>0$

$$\lim_{x\to\infty}\frac{L(c\cdot x)}{L(x)}=1 \qquad (3.4.8)$$

Ex.: $L(x)=\log x$, since $\frac{\log(c\cdot x)}{\log x}=\frac{\log c+\log x}{\log x}\xrightarrow[x\to\infty]{}1$ for all $c>0$

$\longrightarrow$ Th. 3.4.2 (Representation theorem for slowly varying functions)

Let L be a slowly varying function. Then there ex. positive funct. c and ε s.t.

$$L(x)=c(x)\exp\left\{\int_{x_0}^{x}\frac{\varepsilon(t)}{t}\,dt\right\},\quad x\geq x_0 \qquad (3.4.9)$$

for some $x_0>0$, where $\varepsilon(t)\xrightarrow[t\to\infty]{}0$ and $c(t)\xrightarrow[t\to\infty]{}c_0$ for some constant $c_0>0$.

Rem. 3.4.3: It follows from (3.4.9) that

$$\lim_{x\to\infty}\frac{L(x)}{x^\delta}=0 \quad\text{and}\quad \lim_{x\to\infty}x^\delta L(x)=\infty \qquad (3.4.10)$$

for all $\delta>0$

$\longrightarrow$ L "small" compared to the power function x^δ

$\longrightarrow$ Def. 3.4.4 (Regularly varying functions and r.v.'s)

Let L be slowly varying. Then

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(i) for any $\delta \in \mathbb{R}$, a function $f:(0,\infty) \longrightarrow (0,\infty)$ given by
$$f(x) = x^{\delta} L(x), \; x>0 \qquad (3.4.11)$$
is called regularly varying with index δ

(ii) a positive r.v. X is said to be regularly varying with index $\alpha \geq 0$, if
$$\overline{F}_X(x) \overset{\text{def.}}{=} P(X>x) = x^{-\alpha} L(x), \; x>0 \quad (3.4.12)$$

Rem. 3.4.5 (alternative definition)
$$\lim_{x \to \infty} \frac{f(c \cdot x)}{f(x)} = c^{\delta} \text{ for all } c>0 \quad (3.4.13)$$

Rem. (i) (3.4.12) $\longrightarrow$ "small" deviation of $\overline{F}_X$ from the exact power law

$\Longrightarrow$ for $\alpha>0$ the r.v. X has a heavy (right) tail

(ii) X regularly varying with index $\alpha>0$
$$\overset{(3.4.9)}{\Longrightarrow} E[X^{\delta}] = \begin{cases} \infty, & \text{if } \delta > \alpha \\ <\infty, & \text{if } \delta < \alpha \end{cases} \quad (3.4.14)$$

(iii) Pareto, Burr and log-Gamma distr. X are regularly varying with an index $\alpha>0$

$\longrightarrow$ natural question: What is the impact of "dangerous" claims, that is of regularly varying claim size variables with index $\alpha>0$ on allocated claims?

$\longrightarrow$ Prop. 3.4.6: Let $X_1, \dots, X_n$ i.i.d regularly varying r.v.'s with index $\alpha>0$ and $X_i \sim F$. Define $S_n = \sum_{i=1}^{n} X_i$ and $M_n = \max(X_1, \dots, X_n)$. Then
$$\lim_{x \to \infty} \frac{P(S_n>x)}{P(M_n>x)} = 1 \text{ for all } n \geq 2 \quad (3.4.15)$$

Proof: See T. Mikosch.

$\longrightarrow$ for "dangerous" or more mathematically regularly varying claims X_i with index $\alpha>0$ the tail distr. of S_n and M_n are asymptotically the same