$\longrightarrow$ (3.4.15) gives rise to the following definition of subexponentiality (or heavy-tailedness) of distr.

Def. 3.4.7 (Subexponential r.v.)
Let X be a positive r.v. with distr. function F s.t. $F(x)<1$ for $x\in(0,\infty)$. Further, let X_i, $i\geq 1$ be an i.i.d.-sequence of r.v.'s with $X_i\sim F$.
Then X and F are called subexponential, if
$$\lim_{x\to\infty}\frac{P(S_n>x)}{P(M_n>x)}=1 \quad \text{for all } n\geq 2$$
The class of such distr. is denoted by S.

Rem. 3.4.8 (equivalent definitions)
(3.4.15) $\Longleftrightarrow$ (3.4.15) holds for some $n\geq 2$ $\Longleftrightarrow$
$$\lim_{x\to\infty}\frac{P(S_n>x)}{\overline{F}(x)}=n \qquad (3.4.16)$$
for all $n\geq 1$.

L. 3.4.9 (Basic properties of $F\in S$)
(i) Let $F\in S$, then for all $y>0$:
$$\lim_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)}=1 \qquad (3.4.17)$$
(ii) Suppose (3.4.17) holds for all $y>0$. Then for all $\varepsilon>0$
$$e^{\varepsilon\cdot x}\,\overline{F}(x)\xrightarrow[x\to\infty]{}\infty \qquad (3.4.18)$$
(iii) Let $F\in S$. Then for all $\varepsilon>0$ there ex. a $K=K(\varepsilon)<\infty$ s.t. for all $n\geq 2$
$$\frac{P(S_n>x)}{\overline{F}(x)}\leq K(1+\varepsilon)^n, \quad x\geq 0 \qquad (3.4.19)$$

Rem. 3.4.10
(i) Let $F=F_X\in S$. Then
$$E[e^{\varepsilon X}]\geq E[\mathbb{1}_{(y,\infty)}(X)\cdot e^{\varepsilon\cdot X}]\geq \overbrace{e^{\varepsilon y}\underbrace{E[\mathbb{1}_{(y,\infty)}(X)]}_{=P(X>y)=\overline{F}(y)}}^{\xrightarrow{(3.4.18)}\infty \text{ for } y\to\infty}$$
$\Rightarrow$ $E[e^{\varepsilon X}]=\infty$ for all $\varepsilon>0$
$\longrightarrow$ (3.4.18) justifies the notion "subexponential", since $\overline{F}(x)$ decays slower to zero than $e^{-\varepsilon x}$ for any $\varepsilon>0$

(30) (ii) If $F = F_X \in S$ it follows that

$$1 \xleftarrow[\infty \leftarrow x]{(3.4.17)} \frac{P(X > x + \overset{>0}{y})}{P(X > x)} = \frac{P(X > x+y, X > x)}{P(X > x)} \overset{\text{def.}}{=} P(X > x+y \mid X > x)$$

$\longrightarrow$ once X exceeds a high threshold, it exceeds very likely the even larger value $x+y$

(31) In order to find a Cramér ruin bound for large claims, i.e. subexponential claims, we need the following important representation of non-ruin probabilities $\varphi(u)$:

Prop. 3.4.11 (Representation of $\varphi(u)$ as a compound geom. probab.)
Assume the Cramér-Lundberg model with $E[X_1] < \infty$ and NPC. Further, require that F_{X_1} has a prob. density and that $X_{I,j}, j \geq 1$ is an i.i.d. sequence of r.v.'s with common prob. distr. given by

$$F_{X_{I,j}}(x) \overset{def}{=} P(X_{I,j} \leq x) = \int_0^x \frac{1}{E[X_1]} P(X_1 > y)\,dy$$

(integrated tail distribution) (3.4.20)

prob. density, since $\int_0^\infty P(X_1 > y)\,dy = E[X_1]$

Then φ defined by

$$\varphi(u) = \frac{\rho}{1+\rho}\Big[1 + \sum_{n \geq 1} (1+\rho)^{-n} P(X_{I,1} + \dots + X_{I,n} \leq u)\Big], \quad u \geq 0$$

(3.4.21)

solves the integral equation (3.3.2).
Furthermore, φ given by (3.4.21) is the unique solution to the fundamental integral eq. for non-ruin prob. (3.3.2) in the class of functions

$$\mathcal{G} \overset{def}{=} \{G: \mathbb{R} \to [0,\infty) : G \text{ non-decreasing, bounded, right-continuous, } G(x) = 0 \text{ for } x < 0\}$$

Proof: Set $q = (1+\rho)^{-1}$ and $p = 1 - q = \rho(1+\rho)^{-1}$

1. Existence: Recall (dobbel-forventning):
X, Y indep. r.v.'s, f function

$$\Rightarrow E[f(X,Y)] = E\big[(E[f(X,y)])\big|_{y=Y}\big] \quad (*)$$

Further, recall: Y with prob. density f_Y, g function

$$\Rightarrow E[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\,dy \quad (**)$$

Here: Choose $f(x,y) = \mathbb{1}_{\{y+x \leq u\}}(x,y)$, $Y = X_{I,1}$, $X = X_{I,2} + \dots + X_{I,n}$ (indep.)

$$\overset{(*)}{\Longrightarrow} P(X_{I,1} + \dots + X_{I,n} \leq u) = E[f(X,Y)] =$$

$E[\mathbb{1}_A] = P(A)$

(32) $E[\underbrace{(E[f(X_1,Y)])}_{=:g(y)}|_{y=Y}] \overset{(**)}{=} \int_{-\infty}^{\infty} g(y)\, \underbrace{f_{Y_1}(y)}_{=P(X_1>y)/E[X_1]}\, dy$

$= \int_0^\infty \overset{g(y)}{P(y+X_{\mathcal{J},2}+\ldots+X_{\mathcal{J},n}\le u)} \cdot \frac{P(X_1>y)}{E[X_1]}\, dy$

$= \int_0^u P(y+X_{\mathcal{J},2}+\ldots+X_{\mathcal{J},n}\le u)\cdot \frac{P(X_1>y)}{E[X_1]}\, dy$

$\overset{(3.4.21)}{\Longrightarrow}$ $\varphi(u) \overset{\text{def.}}{=} p + p\sum_{n\ge 1} q^n P(X_{\mathcal{J},1}+\ldots+X_{\mathcal{J},n}\le u)$

$= p + qp\,[F_{X_1,\mathcal{J}}(u) + \sum_{n\ge 2} q^{n-1} P(X_{\mathcal{J},1}+\ldots+X_{\mathcal{J},n}\le u))$

$= p+qp\,[F_{X_1,\mathcal{J}}(u) + \sum_{n\ge 2} q^{n-1}\int_0^u \underbrace{P(y+X_{\mathcal{J},2}+\ldots+X_{\mathcal{J},n}\le u)}_{=P(\underbrace{X_{\mathcal{J},2}+\ldots+X_{\mathcal{J},n}}_{\sim X_{\mathcal{J},1}+\ldots+X_{\mathcal{J},n-1}}\le u-y)}\frac{P(X_1>y)}{E[X_1]}\, dy$ $\leftarrow X_{\mathcal{J},j},\ j\ge 1$ i.i.d.

$\overset{\text{def}}{=} \int_0^u \frac{P(X_1>y)}{E[X_1]}\, dy$

$= p+q\,\underbrace{p\,[F_{X_1,\mathcal{J}}(u) + \sum_{n\ge 2} q^{n-1}}_{\text{more}}\int_0^u P(X_{\mathcal{J},1}+\ldots+X_{\mathcal{J},n-1}\le u-y)\frac{P(X_1>y)}{E[X_1]}\, dy$

$= p+q\int_0^u \varphi(u-y)\cdot\frac{P(X_1>y)}{E[X_1]}\, dy \quad \Longrightarrow$ existence.

(ii) Uniqueness: See T. Mikosch.

Rem. 3.4.12: M r.v. geometrically distributed $\Longleftrightarrow$

$p_n \overset{\text{def.}}{=} P(M=n) = p\cdot q^n,\ n\ge 0$ for some $p = 1-q\in(0,1)$

Let $X_i,\ i\ge 1$ be i.i.d. r.v.'s being indep. of M

$\longrightarrow$ r.v. of the form

$$S_M \overset{\text{def.}}{=} \sum_{i=1}^M X_i$$

is called compound geometrically distributed r.v.

$\longrightarrow F_{S_M}(u) = P(S_M\le x) = E[\mathbb{1}_{\{\sum_{i=1}^M X_i\le x\}}]$

$= E[\mathbb{1}_{\{\sum_{i=1}^M X_i\le x\}}\underbrace{(\sum_{n\ge 0}\mathbb{1}_{\{M=n\}})}_{=1}]$

$= \sum_{n\ge 0} E[\mathbb{1}_{\{\sum_{i=1}^{M} X_i\le x\}}\mathbb{1}_{\{M=n\}}]$ ($M=n$ on) $= \sum_{n\ge 0} E[\mathbb{1}_{\{\sum_{i=1}^n X_i\le x\}}\mathbb{1}_{\{M=n\}}]$ indep.

$E[X\cdot Y] = E[X]\cdot E[Y]$, if X, Y indep.

$= \sum_{n\ge 0}\underbrace{E[\mathbb{1}_{\{M=n\}}]}_{=p_n = p\cdot q^n}\cdot \overbrace{E[\mathbb{1}_{\{\sum_{i=1}^n X_i\le x\}}]}^{=P(X_1+\ldots+X_n\le x)} \quad (3.4.22)$