Non-Life Insurance Mathematics (STK4540)
Prøveeksamen

Problem 1 Consider the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 09/30/1990 until 10/10/1990:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
10/01/1990	1.023102	10/04/1990	2.145215
10/01/1990	4.702970	10/04/1990	1.155116
10/02/1990	3.135314	10/07/1990	1.056106
10/03/1990	1.864686	10/08/1990	144.657591
10/03/1990	2.037954	10/10/1990	28.630363

The fire losses in the above table are given in millions of Danish Krone. Assume that the dynamics of the total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$ is described by the Crámer-Lundberg model.

(i) Compute the maximum-likelihood estimator $\widehat{\lambda}$ of the jump intensity of the claim number process $N(t)$.

(ii) Denote by $F^* = F^*_{X_1}$ the real claim size distribution. Assume that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, require that F^* has a probability density and that its integrated tail distribution $F^*_{X_1,I}$ is subexponential.

Give a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 30\%$ and the initial capital $u = 100$ (in millions of Danish Krone) by approximating F^* by the empirical distribution function with respect to the fire insurance data above.

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving on day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^{n} X_i^{(m)}$ arriving on day m.

Problem 2 Assume the heterogeneity model and require that in the i-th policy of a car insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ.

(i) Suppose that θ is Gamma distributed with parameters $\gamma, \beta > 0$.

Calculate the linear Bayes estimator $\widehat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X\,|\theta]$ and its corresponding risk for the claim number data $X_1 = 4$, $X_5 = 7$, $X_7 = 3, X_{10} = 1$ and $X_j = 0$ for $j = 2, 3, 4, 6, 8, 9$ (in years) and the parameters $\gamma = 2.9, \beta = 2$.

(ii) Assume now that $\theta \sim U(0, 1)$ (uniform distribution on $(0, 1)$).

Compute $\widehat{\mu}_{LB}$ for the same data as in (i).

Hint: Recall that $E[\theta] = \frac{\gamma}{\beta}$, $Var[\theta] = \frac{\gamma}{\beta^2}$, if $\theta \sim \Gamma(\gamma, \beta)$.

Problem 3 Let us have a look at the heterogeneity model for the i-th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$ (Gamma distribution) the claim sizes $X_1, ..., X_n$ in the policy are assumed to be Pareto distributed with parameters (λ, θ), that is

$$P(X_i > x\,|\theta) = (\lambda/x)^{\theta}, x > \lambda.$$

A reinsurance company takes into account only the values X_i exceeding a known high treshold K. They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company is interested in estimating $p(\theta) := P(X_1 > K\,|\theta)$.

Calculate the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1\,|\theta]$ based on the observed data $Y_1, ..., Y_n$, when $\lambda = 1, K = 10$, $X_1 = 10$, $X_5 = 20$, $X_{10} = 50$ $X_j = 1$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8, 9$ (in years). Suppose that $\beta = \gamma = 2$.

Problem 4 Consider the i-th policy in a heterogeneity model. Assume that the claim size variables $X_j, j \geq 1$ given θ are log-normally distributed, that is $X_j = \exp(\theta + \tau Z_j), j \geq 1$. Here θ is normally distributed with mean μ and variance σ^2, $\tau > 0$ is a constant and $Z_j, j \geq 1$ is an $i.i.d.$-sequence of standard normally distributed random variables being independent of θ.

Compute the linear Bayes estimator with respect to the premium of an insurance and for the claim size data $X_1 = 20$, $X_5 = 100$, $X_j = 1$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8$ (in years) and the parameters $\mu = 3, \sigma^2 = \tau = \frac{1}{2}$.

Problem 5 Assume the heterogeneity model for an insurance company with several policies. We will focus on one given policy. Suppose a claim history of this policy is described by the sequence $X_1, ..., X_n, n \geq 1$ of random variables, which are exponentially distributed given θ. Here θ is the heterogeneity parameter following a Gamma distribution with shape parameter α and scale parameter $\beta > 0$.

(i) Compute the Bayes estimator of the net premium $\mu(\theta) = E[X_1\,|\theta]$ and its corresponding risk for $\alpha = 3, \beta = 1$ and data $X_3 = 10, X_5 = 20, X_7 = 5, X_{10} = 30, X_j = 1, j = 1, 2, 4, 6, 8, 9$ (in 1000 NOK per year).

(ii) Calculate the corresponding linear Bayes estimator with respect to (i).