Non-Life Insurance Mathematics (STK4540)
Solutions to the exam

Problem 1 By Theorem 6.3.1 in Mikosch we have that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

By Exercises 5, Prob. 2 we know that

1.

$$\begin{aligned}\mu &= E[p(\theta)] = E[P(X_1 > K\,|\theta)] = E[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} \approx 0.098.\end{aligned}$$

2.

$$\begin{aligned}\lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{2\gamma} \approx 0.0174.\end{aligned}$$

3.

$$\begin{aligned}\varphi &= E[Var[X_1\,|\theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} \approx 0.0711.\end{aligned}$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.0174}{0.0711 + 10 \cdot 0.0174} \approx 0.71 \text{ and } \overline{Y} = 0.1.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y} \approx 9.94\%$$

is the estimated probability for $X_1 > 50000$ NOK given θ.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.0050.$$

Problem 2 (i) Since the claim numbers are modelled by a Poisson process, the inter-arrival times $W_i, i \geq 1$ are *i.i.d.* with common distribution $W_1 \sim Exp(\lambda)$. So by Exercises 1, Problem 1 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} W_i}.$$

The observed arrival times are $W_1 = 2$ (the day 07/31/1989 is excluded), $W_2 = 1, W_3 = 1, W_4 = 1, W_5 = 2, W_6 = 1, W_7 = 1$. So $n = 7$ (sample size) and we get that

$$\widehat{\lambda} = \frac{7}{9} \approx 0.778.$$

(ii) We know that

$$\Psi(u) \sim \rho^{-1}\overline{F}^*_{X_1,I}(u)$$

for $u \longrightarrow \infty$ (Th. 3.4.13).

Next we want to approximate

$$F^*_{X_1,I}(u) = \frac{1}{E^*[X_1]} \int_0^u P^*(X_1 > y)dy$$

by means of the empirical distribution function F_n.

So we see that

$$E^*[X_1] \approx \frac{1}{n}\sum_{i=1}^n X_i \overset{n=7}{=} 26.809.$$

On the other hand,

$$P^*(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned} \int_0^u P^*(X_1 &> y)dy \approx \int_0^u (1 - \frac{1}{n}\sum_{i=1}^n 1_{(-\infty,y]}(X_i))dy \\ &= u - \frac{1}{n}\sum_{i=1}^n (\max(u, X_i) - X_i) \approx 22.532. \end{aligned}$$

Using the latter, we find that

$$\Psi(140) \approx 63.814\%.$$

Problem 3 (i) We apply Theorem 6.3.1 in Mikosch and get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^n X_j.$$

Here:

1.
$$\mu = E[\mu(\theta)] = E[\theta] \overset{\text{hint}}{=} \frac{2.7}{1.5} = 1.8.$$

2.
$$\lambda = Var[\mu(\theta)] = Var[\theta] \overset{\text{hint}}{=} \frac{2.7}{(1.5)^2} = 1.2.$$

3.
$$\varphi = E[Var[X_1 \,|\theta]] = E[\theta] \overset{\text{hint}}{=} \frac{2.7}{1.5} = 1.8.$$

Thus

$$w \overset{n=10}{=} \frac{10 \cdot 1.2}{1.8 + 10 \cdot 1.2} = \frac{12}{13.8} \approx 0.860 \text{ and } \overline{X} = 0.8.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1 - w)\mu + w\overline{X} \approx 0.94.$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1 - w)\lambda \approx 0.168.$$

(ii) We now assume in (i) that $\theta = \exp(Z)$ for $Z \sim \mathcal{N}(0, 1)$. Then

1.

$$\mu = E[\mu(\theta)] = E[\exp(Z)] = e^{\frac{1}{2}} \approx 1.649.$$

2.

$$\begin{aligned} \lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[e^{2Z}] - \mu^2 = e^{\frac{1}{2} \cdot 4} - \mu^2 \\ &\approx 4.670. \end{aligned}$$

3.

$$\varphi = E[Var[X_1 \,|\theta]] = E[\theta] \approx 1.649.$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 4.67}{1.649 + 10 \cdot 4.67} \approx 0.966 \text{ and } \overline{X} = 0.8.$$

Hence

$$\widehat{\mu}_{LB} = (1 - w)\mu + w\overline{X} \approx 0.829.$$

Problem 4 (i) Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot \ldots \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}^{(m)}_j = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

So

$$\begin{aligned} \widehat{f}^{(m)}_0 &= \frac{\sum_{i=2013}^{2016} S_{i,i+1}}{\sum_{i=2013}^{2016} S_{i,i}} = \frac{490629}{323661} \approx 1.516, \\ \widehat{f}^{(m)}_1 &= \frac{\sum_{i=2013}^{2015} S_{i,i+2}}{\sum_{i=2013}^{2015} S_{i,i+1}} = \frac{412845}{340153} \approx 1.214, \\ \widehat{f}^{(m)}_2 &= \frac{\sum_{i=2013}^{2014} S_{i,i+3}}{\sum_{i=2013}^{2014} S_{i,i+2}} = \frac{291229}{254436} \approx 1.145, \end{aligned}$$

$$\widehat{f}_3^{(m)} = \frac{\sum_{i=2013}^{2013} S_{i,i+4}}{\sum_{i=2013}^{2013} S_{i,i+3}} \approx 1.039.$$

Hence, we obtain the following predictors in the run-off triangle table:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2013	65285	88495	120096	138346	143682
2014	73173	114299	134340	152883	158845
2015	87135	137359	158409	181378	188452
2016	98068	150476	182678	209166	317096
2017	138982	210697	255786	292875	304297

(ii) Using those predictors in (i), we get the technical provision for 2019, i.e.

$$(188452 - 181378) + (209166 - 182678) + (255786 - 210697) = 78651.$$

and that of 2021, i.e.

$$304297 - 292875 = 11422.$$

Problem 5 (i) Define $M_n = \max_{i=1,\dots,n} X_i$. Then we obtain by the definition of the random times $T_n, n \geq 1$, the convention $\max_{j=2,\dots,1} X_j := 0$, "dobbel-forventning" and the geometrical sum that

$$\begin{aligned}
P(X_{T_2} &> x, X_1 < x, T_2 < \infty) \\
&= P(X_{T_2} > x, \max_{j=2,\dots,T_2-1} X_j \leq X_1, X_1 < x, T_2 < \infty) \\
&= \sum_{n\geq 2} P(X_{T_2} > x, \max_{j=2,\dots,T_2-1} X_j \leq X_1, X_1 < x, T_2 = n) \\
&= \sum_{n\geq 2} P(X_n > x, \max_{j=2,\dots,n-1} X_j \leq X_1, X_1 < x, T_2 = n) \\
&= \sum_{n\geq 2} P(X_n > x, \max_{j=2,\dots,n-1} X_j \leq X_1, X_1 < x) \\
\overset{X_i \text{ i.i.d.}}{=} \sum_{n\geq 2} P(X_n &> x) P(\max_{j=2,\dots,n-1} X_j \leq X_1, X_1 < x).
\end{aligned}$$

But

$$\begin{aligned}
\sum_{n\geq 2} P(X_n &> x) P(\max_{j=2,\dots,n-1} X_j \leq X_1, X_1 < x) \\
&= P(X_1 > x) \sum_{n\geq 2} E[(F_{X_1}(X_1))^{n-2} \mathbf{1}_{(0,x)}(X_1)] \\
&= P(X_1 > x) E[\sum_{n\geq 2} (F_{X_1}(X_1))^{n-2} \mathbf{1}_{(0,x)}(X_1)] \\
&= P(X_1 > x) E[\frac{1}{1 - F_{X_1}(X_1)} \mathbf{1}_{(0,x)}(X_1)] = P(X_1 > x) \int_0^x \frac{1}{1 - F_{X_1}(y)} g(y) dy.
\end{aligned}$$

So for Pareto distributed X_1 with parameters $\gamma = \theta = 1$, we find that

$$\begin{aligned} P(X_{T_2} &> x, X_1 < x, T_2 < \infty) = \frac{1}{x}\int_1^x \frac{1}{\frac{1}{y}}\frac{1}{y^2}dy \\ &= \frac{\log(x)}{x}, \text{ if } x > 1 \text{ and zero else.} \end{aligned}$$

(ii) The definition of Y_n gives

$$Y_n = 1 + \sum_{k=2}^{n} \mathbf{1}_{\{X_k > M_{k-1}\}}$$

for $n \geq 2$. The latter implies that

$$E[Y_n] = 1 + \sum_{k=2}^{n} P(X_k > M_{k-1}).$$

Since the fire losses X_i are *i.i.d* with a continuous common distribution we find that

$$\begin{aligned} 1 &= P(X_1 \geq X_2) + P(X_2 > X_1) \\ P(X_1 &> X_2) + P(X_2 > X_1) = 2P(X_2 > M_1). \end{aligned}$$

So $P(X_2 > M_1) = \frac{1}{2}$. In the same way we conclude that

$$P(X_k > M_{k-1}) = \frac{1}{k}$$

for $k \geq 3$. On the other hand, if we compute $E[Y_n^2]$ we need to calculate probabilities of the form

$$P(X_{k_1} > M_{k_1 - 1}, X_{k_2} > M_{k_2 - 1})$$

for $k_1 > k_2$. Taking into account all possible permutations we get (by induction) that the latter probability must be

$$(k_2 + 1) \cdot \ldots \cdot (k_1 - 1)\frac{1}{k_1!}(k_2 - 1)! = \frac{1}{k_1 k_2}$$

Thus we have

$$Var[Y_n] = E[Y_n^2] - (E[Y_n])^2 = \sum_{k=1}^{n}(\frac{1}{k} - \frac{1}{k^2}).$$

Hence

$$Var[Y_7] \approx 1.081.$$

(iii) One checks that

$$\max(0, X - Y) = X - \min(X, Y)$$

and

$$\max(0, Y - X) = Y - \min(X, Y).$$

Taking the expectation, we see that

$$E[\max(0, X - Y)] = E[X] - E[\min(X, Y)] = E[X] - E[Y] = E[X] - E[X] = 0.$$

Therefore, since $\max(0, X - Y) \geq 0$, it follows that

$$\max(0, X - Y) = 0$$

with probability 1. So $X \leq Y$ with probability 1. Similarly, we get $Y \geq X$ with probability 1. Hence $P(X = Y) = 1$, which implies that $Var[X - Y] = 0$.