UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in STK4540 — Non-Life Insurance Mathematics.

Day of examination: December 04, 2020, Final Home Exam.

Examination hours: 15:00 – 19:00.

This problem set consists of 3 pages.

Appendices: None

Permitted aids: See the webpage of the course.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Consider the heterogeneity model for the i-th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$ (Gamma distribution) the claim sizes $X_1, ..., X_n$ in the policy are assumed to be Pareto distributed with parameters (λ, θ), that is

$$P(X_i > x \,|\theta) = (\lambda/x)^\theta, x > \lambda.$$

A reinsurance company takes into account only the values X_i which exceed a known high treshold K. They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company is interested in estimating $p(\theta) := P(X_1 > K \,|\theta)$.

Compute the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1 \,|\theta]$ based on the observations $Y_1, ..., Y_n$, when $X_1 = 20$, $X_5 = 40$, $X_{10} = 100$ and $X_j = 2$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8, 9$ (in years) and the parameters $\lambda = 1.5, \beta = \gamma = 3, K = 50$.

Problem 2. (10 points)

Have a look at the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 07/31/1989 until 08/09/1989:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
08/02/1989	1.270110	08/05/1989	1.439458
08/03/1989	1.784928	08/05/1989	3.130398
08/04/1989	16.088061	08/07/1989	6.287045
08/04/1989	1.439458	08/08/1989	2.777307
08/04/1989	152.413209	08/09/1989	1.033023

The fire losses in the above table are given in millions of Danish Krone. Suppose that the dynamics of the total claim amount $S(t)$ is described by the Crámer-Lundberg model.

(Continued on page 2.)

(i) Calculate the maximum-likelihood estimator $\widehat{\lambda}$ of the jump intensity of the claim number process $N(t)$.
(ii) Let $F^* = F^*_{X_1}$ be the real claim size distribution. Require that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, assume that F^* has a probability density and that its integrated tail distribution $F^*_{X_1,I}$ is subexponential.
Compute a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 25\%$ and the initial capital $u = 140$ (in millions of Danish Krone) by approximating F^* by the empirical distribution function with respect to the fire insurance data above.

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving on day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^n X_i^{(m)}$ arriving on day m.

Problem 3. (10 points)

Consider the heterogeneity model and suppose that in the i-th policy of an insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ.
(i) Require that θ is Gamma distributed with parameters $\gamma, \beta > 0$.
Compute the linear Bayes estimator $\widehat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X\,|\theta]$ and its corresponding risk for the claim number data $X_1 = 1$, $X_4 = 2, X_5 = 1$, $X_7 = 1, X_{10} = 3$ and $X_j = 0$ (in 1000 NOK) for $j = 2, 3, 6, 8, 9$ (in years) and the parameters $\gamma = 2.7, \beta = 1.5$.
(ii) Suppose now that $\theta = \exp(Z)$, where $Z \sim \mathcal{N}(0, 1)$ (normal distribution with mean zero and variance 1).
Calculate $\widehat{\mu}_{LB}$ for the same data as in (i).

Hint: $E[\theta] = \frac{\gamma}{\beta}$, $Var[\theta] = \frac{\gamma}{\beta^2}$, if $\theta \sim \Gamma(\gamma, \beta)$.

Problem 4. (10 points)

Assume Mack's model. The following run-off triangle in the figure shows the cumulative paid claims:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2013	65285	88495	120096	138346	143682
2014	73173	114299	134340	152883	
2015	87135	137359	158409		
2016	98068	150476			
2017	138982				

(i) Compute the predictors $\widehat{S}_{i,m+k}$ of the total claim amounts

$$S_{2014,m+1},$$
$$S_{2015,m+1}, S_{2015,m+2},$$
$$S_{2016,m+1}, ..., S_{2016,m+3},$$
$$S_{2017,m+1}, ..., S_{2017,m+4}.$$

(Continued on page 3.)

for $m = 2017$ based on the data of the run-off triangle.
(ii) Compute the technical provisions for the year 2019, that is

$$\sum_{j=0}^{2}(\widehat{S}_{2015+j,2019} - \widehat{S}_{2015+j,2018})$$

and for the year 2021, that is

$$\widehat{S}_{2017,2021} - \widehat{S}_{2017,2020}.$$

Problem 5. (10 points)

Consider an *i.i.d*-sequence $(X_i)_{i\geq 1}$ of wind storm losses. Require that X_1 has a probability density f and that $P(X_1 > x) > 0$ for all $x \geq 0$. Let us now have a look at stochastic time points T_n, $n \geq 1$ defined by

$$T_{n+1} := \inf\{k > T_n : X_k > X_{T_n}\}$$

for $n \geq 1$, where $T_1 := 1$ and $\inf \varnothing := \infty$. Further, define the sequence of random variables given by

$$Y_n := \#\{1 \leq i \leq n : T_i \leq n\}, n \geq 1,$$

where $\#C$ denotes the number of elements in a set C.

(i) Derive a formula for the probability $P(X_{T_2} > x \text{ and } X_1 < x \text{ and } T_2 < \infty)$ for all $x > 0$. Compute this probability for Pareto distributed X_1 with parameters $\gamma = \theta = 1$, that is

$$P(X_1 \leq x) = \begin{cases} 1 - \frac{1}{x} & , \text{if } x > 1 \\ 0 & \text{else} \end{cases}$$

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(ii) Find a formula for $Var[Y_n]$ and calculate $Var[Y_7]$.

(iii) Assume that the random variables X and Y model wind storm losses. Suppose that $E[X^2], E[Y^2] < \infty$ and that $X \stackrel{d}{=} Y$ and $Y \stackrel{d}{=} min(X, Y)$. Calculate $Var[X - Y]$.

Hint: For random variables Z_1, Z_2 the notation $Z_1 \stackrel{d}{=} Z_2$ means that Z_1 and Z_2 have the same distribution.

End