Non-Life Insurance Mathematics (STK4540)
Solutions to the exam

Problem 1 By Theorem 6.3.1 in Mikosch we have that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

By Exercises 5, Prob. 2 we know that
1.

$$\begin{aligned}\mu &= E[p(\theta)] = E[P(X_1 > K\,|\theta)] = E[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} \approx 0.129761.\end{aligned}$$

2.

$$\begin{aligned}\lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{2\gamma} \approx 0.0259784.\end{aligned}$$

3.

$$\begin{aligned}\varphi &= E[Var[X_1\,|\theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} \approx 0.0869448.\end{aligned}$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.0259784}{0.0869448 + 10 \cdot 0.0259784} \approx 0.749242 \text{ and } \overline{Y} = 0.1.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y} \approx 10.75\%$$

is the estimated probability for $X_1 > 40000$ NOK given θ.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.00651427.$$

Problem 2 (i) We apply Theorem 6.3.1 in Mikosch and get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Here:

1.
$$\mu = E[\mu(\theta)] = E[\theta] \overset{\text{hint}}{=} \frac{2.5}{1} = 2.5.$$

2.
$$\lambda = Var[\mu(\theta)] = Var[\theta] \overset{\text{hint}}{=} \frac{2.5}{(1)^2} = 2.5.$$

3.
$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] \overset{\text{hint}}{=} \frac{2.5}{1} = 2.5.$$

Thus

$$w \overset{n=10}{=} \frac{10 \cdot 2.5}{2.5 + 10 \cdot 2.5} \approx 0.909091 \text{ and } \overline{X} = 1.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 1.14.$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.227273.$$

(ii) We now assume in (i) that $\theta = \exp(Z)$ for $Z \sim U(0,1)$. Then

1.
$$\mu = E[\mu(\theta)] = E[\exp(Z)] = \int_0^1 \exp(y)dy = e - 1 \approx 1.71828.$$

$$\begin{aligned} \lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\ &= E[e^{2Z}] - \mu^2 = \frac{1}{2}(e^2 - 1) - \mu^2 \\ &\approx 0.242036. \end{aligned}$$

3.
$$\varphi = E[Var[X_1\,|\theta]] = E[\theta] \approx 1.71828.$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.242036}{1.71828. + 10 \cdot 0.242036} \approx 0.58482 \text{ and } \overline{X} = 1.$$

Hence

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 1.3.$$

Problem 3 (i) Since the claim numbers are modelled by a Poisson process, the inter-arrival times $W_i, i \geq 1$ are *i.i.d.* with common distribution $W_1 \sim Exp(\lambda)$. So by Exercises 1, Problem 1 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} W_i}.$$

The observed arrival times are $W_1 = 2$ (the day 07/31/1983 is excluded), $W_2 = 3, W_3 = 4, W_4 = 7, W_5 = 1, W_6 = 5, W_7 = 1$. So $n = 7$ (sample size) and we get that

$$\widehat{\lambda} = \frac{7}{23} \approx 0.30435.$$

(ii) We know that

$$\Psi(u) \sim \rho^{-1}\overline{F}^{*}_{X_1,I}(u)$$

for $u \longrightarrow \infty$ (Th. 3.4.13).

Next we want to approximate

$$F^{*}_{X_1,I}(u) = \frac{1}{E^{*}[X_1]} \int_0^u P^{*}(X_1 > y)dy$$

by means of the empirical distribution function F_n.

So we see that

$$E^{*}[X_1] \approx \frac{1}{n}\sum_{i=1}^{n} X_i \overset{n=7}{=} 3.28213.$$

On the other hand,

$$P^{*}(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned} \int_0^u P^{*}(X_1 &> y)dy \approx \int_0^u (1 - \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,y]}(X_i))dy \\ &= u - \frac{1}{n}\sum_{i=1}^{n}(\max(u, X_i) - X_i) \approx 2.8028. \end{aligned}$$

Using the latter, we find that

$$\Psi(4) \approx 73.0215\%.$$

Problem 4 (i) Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot \ldots \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}^{(m)}_j = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

So

$$\widehat{f}^{(m)}_0 = \frac{\sum_{i=2014}^{2017} S_{i,i+1}}{\sum_{i=2014}^{2017} S_{i,i}} = \frac{455427}{326660} \approx 1.39419,$$

$$\widehat{f}^{(m)}_1 = \frac{\sum_{i=2014}^{2016} S_{i,i+2}}{\sum_{i=2014}^{2016} S_{i,i+1}} = \frac{396839}{315154} \approx 1.25919,$$

$$\widehat{f}_2^{(m)} = \frac{\sum_{i=2014}^{2015} S_{i,i+3}}{\sum_{i=2013}^{2014} S_{i,i+2}} = \frac{283155}{243437} \approx 1.16316,$$

$$\widehat{f}_3^{(m)} = \frac{\sum_{i=2014}^{2014} S_{i,i+4}}{\sum_{i=2014}^{2014} S_{i,i+3}} \approx 1.06254.$$

Hence, we obtain the following predictors in the run-off triangle table:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2014	64275	83475	119096	132344	140621
2015	75172	104222	124341	150811	160243.
2016	88142	127457	153402	178430	189590.
2017	99071	140273	176630	205449	218298
2018	139881	195021	245569	285635	303499

(ii) Using those predictors in (i), we get the technical provision for 2020, i.e.

$$\sum_{j=0}^{2} (\widehat{S}_{2016+j,2020} - \widehat{S}_{2016+j,2019})$$

$$(189590 - 178430) + (205449 - 176630) + (245569 - 195021) = 90527$$

and that of 2022, i.e.

$$303499 - 285635 = 17864$$

Problem 5 (i) Consider first the case $n = 2$: Define $L(x_1, x_2) = (\frac{x_2}{2}, \frac{x_2}{2} + \frac{x_1}{1}) = (\frac{x_2}{2}, \frac{x_2}{2} + x_1)$. Then one observes that the function L has an inverse L^{-1} given by $L^{-1}(y_1, y_2) = (y_2 - y_1, 2y_1)$. So we get

$$\begin{aligned}
P(X_{(1)} &\leq x_1, X_{(2)} \leq x_2) = \int_0^{x_1} \int_0^{x_2} f_{X_{(1)},X_{(2)}}(y_1, y_2) dy_1 dy_2 \\
&\overset{\text{Hint}}{=} \int_0^{x_1} \int_0^{x_2} 2! f_{X_1}(y_1) f_{X_2}(y_2) 1_{\{y_2 > y_1\}} dy_1 dy_2 \\
&\int_0^{x_1} \int_0^{x_2} 2\lambda e^{-\lambda(y_2 - y_1)} \lambda e^{-\lambda(2y_1)} 1_{\{y_2 > y_1\}} dy_1 dy_2 \\
&\overset{X_1, X_2 \text{ indep.}}{=} \int_0^{x_1} \int_0^{x_2} 2 \cdot f_{X_1,X_2}(L^{-1}(y_1, y_2)) dy_1 dy_2 \\
&\overset{\text{substitution.:}(u_1,u_2) = L^{-1}(y_1,y_2)}{=} \int_{L^{-1}([0,x_1]\times[0,x_2])} f_{X_1,X_2}(u_1, u_2) du_1 du_2 \\
&= P((X_1, X_2) \in L^{-1}([0, x_1] \times [0, x_2])) \\
&= P(L(X_1, X_2) \in [0, x_1] \times [0, x_2]) = P(\frac{X_2}{2} \leq x_1, \frac{X_2}{2} + \frac{X_1}{1} \leq x_2).
\end{aligned}$$

So

$$(X_{(1)}, X_{(2)}) \overset{d}{=} (\frac{X_2}{2}, \frac{X_2}{2} + \frac{X_1}{1}).$$

In the same way one also shows more generally that

$$(X_{(1)},...,X_{(n)}) \stackrel{d}{=} (\frac{X_n}{n}, \frac{X_n}{n} + \frac{X_{n-1}}{n-1},..., \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + ... + \frac{X_2}{2}, \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + ... + \frac{X_1}{1}). \quad (1)$$

Define

$$G(x_1,...,x_n) = \sum_{i=1}^{k-1} x_{n-i+1} - (k-1) \cdot x_{n-k+1}.$$

Then it follows from (1) for all $n \geq k \geq 2$ that

$$\begin{aligned} \sum_{i=1}^{n} (X_{(n-i+1)} - X_{(n-k+1)})_+ &= \sum_{i=1}^{k-1} X_{(n-i+1)} - (k-1) \cdot X_{(n-k+1)} = G(X_{(1)},...,X_{(n)}) \\ \stackrel{d}{=} G(\frac{X_n}{n},..., \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + ... + \frac{X_1}{1}) &= X_1 + ... + X_{k-1} \sim \Gamma(k-1,\lambda) \text{ (see Exerc. 1, Prob. 1).} \end{aligned}$$

Hence

$$P(R(t) \leq x \,|N(t) \geq k) = P(X_1 + ... + X_{k-1} \leq x).$$

(ii) So

$$P(R(t) \leq 5 \,|N(t) \geq 5) = 1 - e^{-5} \sum_{j=0}^{3} \frac{(5)^j}{j!} \approx 0.735.$$