UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in STK4540 — Non-Life Insurance Mathematics.

Day of examination: December 02, 2021, Final written exam.

Examination hours: 09:00 – 13:00.

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Consider the heterogeneity model for the i-th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$ (Gamma distribution) the claim sizes $X_1, ..., X_n$ in the policy are assumed to be Pareto distributed with parameters (λ, θ), that is

$$P(X_i > x \,|\theta) = (\lambda/x)^\theta, x > \lambda.$$

A reinsurance company takes into account only the values X_i which exceed a known high treshold K. They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company is interested in estimating $p(\theta) := P(X_1 > K\,|\theta)$.

Compute the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1\,|\theta]$ based on the observations $Y_1, ..., Y_n$, when $K = 40$, $\lambda = 1.7$, $X_1 = 15$, $X_5 = 35$, $X_{10} = 90$ and $X_j = 2$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8, 9$ (in years). Further assume that $\beta = \gamma = 2.5$.

Hint: Recall that the Gamma distribution has the probability density

$$f_{\gamma,\beta}(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, x > 0,$$

where $\Gamma(z)$ is the Gamma function.

Problem 2. (10 points)

Consider the heterogeneity model and suppose that in the i-th policy of an insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ.

(i) Require that θ is Gamma distributed with parameters $\gamma, \beta > 0$.

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Compute the linear Bayes estimator $\widehat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X\,|\theta]$ and its corresponding risk for the claim number data $X_1 = 3,\ X_4 = 1, X_5 = 1,\ X_7 = 4, X_{10} = 1$ and $X_j = 0$ for $j = 2, 3, 6, 8, 9$ (in years) and the parameters $\gamma = 2.5, \beta = 1$.
(ii) Suppose now that $\theta = \exp(Z)$, where $Z \sim U(0, 1)$ (uniform distribution).
Calculate $\widehat{\mu}_{LB}$ for the same data as in (i).

Hint: $E[\theta] = \frac{\gamma}{\beta}$, $Var[\theta] = \frac{\gamma}{\beta^2}$, if $\theta \sim \Gamma(\gamma, \beta)$.

Problem 3. (10 points)

Have a look at the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 07/31/1983 until 08/23/1983:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
08/02/1983	3.448276	08/17/1983	1.192436
08/05/1983	4.671858	08/22/1983	1.204093
08/09/1983	1.156841	08/22/1983	2.558398
08/16/1983	1.223582	08/22/1983	1.618465
08/16/1983	4.078930	08/23/1983	1.822024

The fire losses in the above table are given in millions of Danish Krone. Suppose that the dynamics of the total claim amount $S(t)$ is described by the Crámer-Lundberg model.
(i) Calculate the maximum-likelihood estimator $\widehat{\lambda}$ of the jump intensity of the claim number process $N(t)$.
(ii) Let $F^* = F^*_{X_1}$ be the real claim size distribution. Require that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, assume that F^* has a probability density and that its integrated tail distribution $F^*_{X_1,I}$ is subexponential.
Compute a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 20\%$ and the initial capital $u = 4$ (in millions of Danish Krone) by approximating F^* by the empirical distribution function with respect to the fire insurance data above.

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving on day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^{n} X_i^{(m)}$ arriving on day m.

Problem 4. (10 points)

Assume Mack's model. The following *run-off triangle* in the figure shows the cumulative paid claims:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2014	64275	83475	119096	132344	140621
2015	75172	104222	124341	150811	
2016	88142	127457	153402		
2017	99071	140273			
2018	139881				

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(i) Compute the predictors $\widehat{S}_{i,m+k}$ of the total claim amounts

$$\begin{aligned}&S_{2015,m+1},\\&S_{2016,m+1}, S_{2016,m+2},\\&S_{2017,m+1}, ..., S_{2017,m+3},\\&S_{2018,m+1}, ..., S_{2018,m+4}.\end{aligned}$$

for $m = 2018$ based on the data of the run-off triangle.
(ii) Compute the technical provisions for the year 2020, that is

$$\sum_{j=0}^{2}(\widehat{S}_{2016+j,2020} - \widehat{S}_{2016+j,2019})$$

and for the year 2022, that is

$$\widehat{S}_{2018,2022} - \widehat{S}_{2018,2021}.$$

Hint: Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot ... \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}^{(m)}_j = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

Problem 5. (10 points)

Assume the Crámer-Lundberg model for the claim numbers $N(t), t \geq 0$ with jump intensity $\lambda^* > 0$ and the *i.i.d* claim sizes $X_i, i \geq 1$ with $X_1 \sim Exp(\lambda)$. Denote by $X_{(j)}$ the $j-$th smallest claim size of the claim sizes $X_1, ..., X_n$. So $X_{(1)} \leq ... \leq X_{(n)}$. Consider now the claim size amount $R(t)$ at time t given by the losses in excess of the k-th largest claim size, that is consider

$$R(t) = \sum_{i=1}^{N(t)}(X_{(N(t)-i+1)} - X_{(N(t)-k+1)})_+$$

for $N(t) \geq k$ and $k \geq 2$, where $(a)_+ \stackrel{def}{=} \max(a, 0)$ for $a \in \mathbb{R}$.

(i) Derive a formula for the distribution

$$P(R(t) \leq x \,|N(t) \geq k)$$

for $k \geq 2$.
(ii) Calculate $P(R(t) \leq 5 \,|N(t) \geq 5)$ for $\lambda^* = \lambda = 1$ and $t = 10$.

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Hint: Use the fact that the probability density of an order statistics $(X_{(1)}, ..., X_{(n)})$ with respect to $i.i.d$ $X_1, ..., X_n$ with density $f = f_{X_1}$ is given by

$$f_{X_{(1)},...,X_{(n)}}(x_1, ..., x_n) = \begin{cases} n! \prod_{i=1}^{n} f(x_i) & \text{if } x_1 < ... < x_n \\ 0 & \text{else} \end{cases}$$

End