Non-Life Insurance Mathematics (STK4540)
Solutions to the exam

Problem 1 (i) Since the claim numbers are modelled by a Poisson process, the inter-arrival times $W_i, i \geq 1$ are *i.i.d.* with common distribution $W_1 \sim Exp(\lambda)$. So by Exercises 1, Problem 1 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n} W_i}.$$

The observed arrival times are $W_1 = 2$ (the day 06/30/1980 is excluded), $W_2 = 2, W_3 = 3, W_4 = 3, W_5 = 2, W_6 = 2, W_7 = 1, W_8 = 10, W_9 = 1$. So $n = 9$ (sample size) and we get that

$$\widehat{\lambda} = \frac{9}{26} \approx 0.34615.$$

(ii) We know that

$$\Psi(u) \sim \rho^{-1}\overline{F}^{*}_{X_1,I}(u)$$

for $u \longrightarrow \infty$ (Th. 3.4.13).

Next we want to approximate

$$F^{*}_{X_1,I}(u) = \frac{1}{E^{*}[X_1]} \int_0^u P^{*}(X_1 > y)dy$$

by means of the empirical distribution function F_n.

So we see that

$$E^{*}[X_1] \approx \frac{1}{n}\sum_{i=1}^{n} X_i \overset{n=9}{=} 33.2901.$$

On the other hand,

$$P^{*}(X_1 > y) \approx 1 - F_n(y).$$

Hence,

$$\begin{aligned}\int_0^u P^{*}(X_1 &> y)dy \approx \int_0^u (1 - \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,y]}(X_i))dy \\ &= u - \frac{1}{n}\sum_{i=1}^{n}(\max(u, X_i) - X_i) \approx 26.2623.\end{aligned}$$

Using the latter, we find that

$$\Psi(200) \approx 60.317\%.$$

Problem 2 By Theorem 6.3.1 in Mikosch we have that

$$\widehat{\mu}_{LB} = (1 - w)\mu + w\overline{Y},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

By Exercises 5, Prob. 2 we know that

1.

$$\begin{aligned} \mu &= E[p(\theta)] = E[P(X_1 > K\,|\theta)] = E[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} \approx 0.127466. \end{aligned}$$

2.

$$\begin{aligned} \lambda &= Var[p(\theta)] = Var[(\frac{\lambda'}{K})^{\theta}] \\ &= \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{2\gamma} \approx 0.030973. \end{aligned}$$

3.

$$\begin{aligned} \varphi &= E[Var[X_1\,|\theta]] = E[p(\theta)] - E[(p(\theta))^2] \\ &= \left(\frac{\beta}{\beta - \log(\lambda'/K)}\right)^{\gamma} - \left(\frac{\beta}{\beta - 2\log(\lambda'/K)}\right)^{\gamma} \approx 0.0802453. \end{aligned}$$

So

$$w \overset{n=10}{=} \frac{10 \cdot 0.030973}{0.0802453 + 10 \cdot 0.030973} \approx 0.79423 \text{ and } \overline{Y} = 0.2.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{Y} \approx 18.508\%$$

is the estimated probability for $X_1 > 40000$ NOK given θ.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.00637331.$$

Problem 3 (i) We apply Theorem 6.3.1 in Mikosch and get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X},$$

where

$$w = \frac{n\lambda}{\varphi + n\lambda} \text{ and } \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Here:

1.

$$\mu = E[\mu(\theta)] = E[\theta] \overset{\text{hint}}{=} \frac{1.75}{1.3} \approx 1.34615.$$

2.

$$\lambda = Var[\mu(\theta)] = Var[\theta] \stackrel{\text{hint}}{=} \frac{1.75}{(1.3)^2} \approx 1.0355.$$

3.

$$\varphi = E[Var[X_1 | \theta]] = E[\theta] \stackrel{\text{hint}}{=} \frac{1.75}{1.3} \approx 1.34615.$$

Thus

$$w \stackrel{n=10}{=} \frac{10 \cdot 1.0355}{1.34615 + 10 \cdot 1.0355} \approx 0.884956 \text{ and } \overline{X} = 1.2.$$

Therefore, we get that

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} \approx 1.21681.$$

expected claim number (per year) given the observations.

The corresponding risk is given by

$$\rho(\widehat{\mu}_{LB}) = (1-w)\lambda \approx 0.119128.$$

(ii) We now assume in (i) that $\theta = \exp(Z)$ for $Z \sim Exp(3)$. Then

1.

$$\begin{aligned}
\mu &= E[\mu(\theta)] = E[\exp(Z)] = \int_0^\infty \exp(y) 3 \exp(-3y) dy \\
&= 3 \int_0^\infty \exp(-2y) dy = 3(-\frac{1}{2}\exp(-2y)\Big|_{y=0}^{\infty}) = 1.5
\end{aligned}$$

$$\begin{aligned}
\lambda &= Var[\mu(\theta)] = E[(\mu(\theta))^2] - \mu^2 \\
&= E[e^{2Z}] - \mu^2 \\
&= \int_0^\infty \exp(2y) 3 \exp(-3y) dy - \mu^2 \\
&= 3 \int_0^\infty \exp(-y) dy - \mu^2 \\
&= 3(-\exp(-y)|_{y=0}^{\infty}) - \mu^2 \\
&= 3 - \mu^2 = 0.75.
\end{aligned}$$

3.

$$\varphi = E[Var[X_1 | \theta]] = E[\theta] = 1.5.$$

So

$$w \stackrel{n=10}{=} \frac{10 \cdot 0.75}{1.5 + 10 \cdot 0.75} \approx 0.833333 \text{ and } \overline{X} = 1.2.$$

Hence

$$\widehat{\mu}_{LB} = (1-w)\mu + w\overline{X} = 1.25.$$

Problem 4 (i) Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot \ldots \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}_j^{(m)} = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

So

$$\widehat{f}_0^{(m)} = \frac{\sum_{i=2015}^{2018} S_{i,i+1}}{\sum_{i=2015}^{2018} S_{i,i}} \approx 1.39226,$$

$$\widehat{f}_1^{(m)} = \frac{\sum_{i=2015}^{2017} S_{i,i+2}}{\sum_{i=2015}^{2017} S_{i,i+1}} \approx 1.22815,$$

$$\widehat{f}_2^{(m)} = \frac{\sum_{i=2015}^{2016} S_{i,i+3}}{\sum_{i=2015}^{2016} S_{i,i+2}} \approx 1.23144,$$

$$\widehat{f}_3^{(m)} = \frac{\sum_{i=2015}^{2015} S_{i,i+4}}{\sum_{i=2015}^{2015} S_{i,i+3}} \approx 1.08819.$$

Hence, we obtain the following predictors in the run-off triangle table:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2015	66238	87434	113473	135235	147162
2016	70233	99167	117432	149111	162262
2017	89256	123672	150156	184908	201216
2018	101389	145157	178274	219534	238896
2019	142381	198231	243457	299803	326244

(ii) Using those predictors in (i), we get the technical provision for 2021, i.e.

$$\sum_{j=0}^{2}(\widehat{S}_{2017+j,2021} - \widehat{S}_{2017+j,2020})$$

$$(201216 - 184908) + (219534 - 178274) + (243457 - 198231) = 102794$$

and that of 2023, i.e.

$$326244 - 299803 = 26441$$

Problem 5 It can be shown as in Problem 7 of the second mandatory assignment that $S(t), t \geq 0$ has independent and stationary increments. Using the latter property, we find for $0 < t_1 < t_2 < ... < t_n$ that

$$\begin{aligned} & E\left[\exp(-\lambda_1 S(t_1) - ... - \lambda_n S(t_n))\right] \\ = \; & E\left[\exp(-\lambda_1 S(t_1) - \lambda_2(S(t_1) + (S(t_2) - S(t_1))) - ... - \lambda_n(S(t_1) + (S(t_2) - S(t_1)) + ... + (S(t_n) - S(t_{n-1} \right. \\ = \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n)S(t_1) - (\lambda_2 + ... + \lambda_n)(S(t_2) - S(t_1)) - ... - \lambda_n(S(t_n) - S(t_{n-1})))\right] \\ = \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n)S(t_1))\right] \cdot E\left[\exp(-(\lambda_2 + ... + \lambda_n)(S(t_2) - S(t_1)))\right] \cdot ... \cdot E\left[\exp(-\lambda_n(S(t_n) - S(t_{n-1} \right. \\ = \; & E\left[\exp(-(\lambda_1 + ... + \lambda_n)S(t_1))\right] \cdot E\left[\exp(-(\lambda_2 + ... + \lambda_n)(S(t_2 - t_1)))\right] \cdot ... \cdot E\left[\exp(-\lambda_n(S(t_n - t_{n-1})))\right]. \end{aligned}$$

On the other hand, using monotone convergence, we get that

$$
\begin{aligned}
& E\left[\exp(-\lambda_1 S(t_1))\right] \\
= \; & E\left[\exp(-\lambda_1 \sum_{i=1}^{N(t_1)} (X_i - x)_+)\right] \\
= \; & E\left[\exp(-\lambda_1 \sum_{i=1}^{N(t_1)} (X_i - x)_+)(\sum_{k\geq 0} 1_{\{N(t_1)=k\}})\right] \\
= \; & \sum_{k\geq 0} E\left[\exp(-\lambda_1 \sum_{i=1}^{k}(X_i - x)_+)1_{\{N(t_1)=k\}}\right] \\
= \; & \sum_{k\geq 0} E\left[\exp(-\lambda_1 \sum_{i=1}^{k}(X_i - x)_+)\right] P(N(t_1) = k) \\
= \; & \sum_{k\geq 0}(E\left[\exp(-\lambda_1 (X_1 - x)_+)\right])^k \frac{(\lambda t_1)^k}{k!} e^{-\lambda t_1}.
\end{aligned}
$$

By applying the probability density of the exponential distribution, we also find that

$$
\begin{aligned}
& E\left[\exp(-\lambda_1 (X_1 - x)_+)\right] \\
= \; & E\left[1_{\{X_1 \leq x\}} \exp(-\lambda_1 \cdot 0)\right] + E\left[1_{\{X_1 > x\}} \exp(-\lambda_1 (X_1 - x))\right] \\
= \; & P(X_1 \leq x) + e^{\lambda_1 x} E\left[1_{\{X_1 > x\}} \exp(-\lambda_1 X_1)\right] \\
= \; & 1 - e^{-\gamma x} + e^{\lambda_1 x} \int_x^\infty \exp(-\lambda_1 y)\gamma \exp(-\gamma y) dy \\
= \; & 1 - e^{-\gamma x} + \gamma e^{\lambda_1 x} \int_x^\infty \exp(-(\lambda_1 + \gamma) y) dy \\
= \; & 1 - e^{-\gamma x} + \gamma e^{\lambda_1 x} \left(-\frac{1}{(\lambda_1 + \gamma)} \exp(-(\lambda_1 + \gamma) y) \Big|_{y=x}^{\infty} \right) \\
= \; & 1 - e^{-\gamma x} + \gamma e^{\lambda_1 x} \frac{1}{(\lambda_1 + \gamma)} \exp(-(\lambda_1 + \gamma) x) \\
= \; & 1 - e^{-\gamma x} + \frac{\gamma}{(\lambda_1 + \gamma)} e^{-\gamma x} \\
= \; & 1 + (\frac{\gamma}{(\lambda_1 + \gamma)} - 1) e^{-\gamma x}.
\end{aligned}
$$

So

$$
\begin{aligned}
& E\left[\exp(-\lambda_1 S(t_1))\right] \\
= \; & \sum_{k\geq 0}(1 + (\frac{\gamma}{(\lambda_1 + \gamma)} - 1)e^{-\gamma x})^k \frac{(\lambda t_1)^k}{k!} e^{-\lambda t_1} \\
= \; & \exp\left((1 + (\frac{\gamma}{(\lambda_1 + \gamma)} - 1)e^{-\gamma x})\lambda t_1 - \lambda t_1\right) \\
= \; & \exp\left((\frac{\gamma}{(\lambda_1 + \gamma)} - 1)e^{-\gamma x}\lambda t_1\right).
\end{aligned}
$$

On the other hand, consider the total claim amount process

$$\widetilde{S}(t) = \sum_{i=1}^{N^*(t)} X_i,$$

where $N^*(t), t \geq 0$ is a homogeneous Poisson process with intensity $\lambda^* > 0$, which is independent of the claim sizes $X_i, i \geq 1$. Then we can carry out the same calculations as above for $x = 0$ and obtain that

$$\begin{aligned} & E\left[\exp(-\lambda_1 \widetilde{S}(t_1))\right] \\ = \; & \exp\left((\frac{\gamma}{(\lambda_1 + \gamma)} - 1)\lambda^* t_1 \right). \end{aligned}$$

Choose now $\lambda^* = \lambda e^{-\gamma x}$. Then we see that

$$E\left[\exp(-\lambda_1 \widetilde{S}(t_1))\right] = E\left[\exp(-\lambda_1 S(t_1))\right]$$

for all $\lambda_1, t_1 > 0$. Since also $\widetilde{S}(t), t \geq 0$ is a process with independent and stationary increments, it follows from (1) and the hint that

$$(\widetilde{S}(t_1), ..., \widetilde{S}(t_n)) \overset{\text{law}}{=} (S(t_1), ..., S(t_n))$$

for all $0 < t_1 < t_2 < ... < t_n$. The latter (in connection with the Laplace-Stieltjes transform) also implies that

$$(\widetilde{U}(t_1), ..., \widetilde{U}(t_n)) \overset{\text{law}}{=} (U(t_1), ..., U(t_n))$$

for all $0 < t_1 < t_2 < ... < t_n$, where $\widetilde{U}(t) := u + ct - \widetilde{S}(t)$ is another risk process. Hence,

$$\Psi(u) = \widetilde{\Psi}(u),$$

where $\widetilde{\Psi}(u)$ is the ruin probability with respect to $\widetilde{U}(t), t \geq 0$ (which also satisfies the Net Profit condition for $\gamma \geq 1$).

We also know from Problem 3, Exercises 2 that

$$\widetilde{\Psi}(u) = \frac{1}{1+\rho} \exp(-\gamma \frac{\rho}{1+\rho} u).$$

So for $u = 20$, $\gamma = \lambda = 1$, $x = 15$ and $\rho = 0.2$ we get that $\Psi(20) = 0.0297$.