UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in STK4540 — Non-Life Insurance Mathematics.

Day of examination: December 02, 2022, Final written exam.

Examination hours: 09:00 – 13:00.

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

Consider the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 06/30/1980 until 07/26/1980:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
07/02/1980	1.464129	07/12/1980	1.677892
07/04/1980	3.963250	07/14/1980	1.449488
07/04/1980	5.563852	07/15/1980	263.250366
07/07/1980	4.392387	07/25/1980	2.500732
07/10/1980	3.640313	07/26/1980	11.7086

The fire losses in the above table are given in millions of Danish Krone. Assume that the dynamics of the total claim amount $S(t), t \geq 0$ is described by the Crámer-Lundberg model.

(i) Compute the maximum-likelihood estimator $\widehat{\lambda}$ of the intensity of the claim number process $N(t), t \geq 0$.

(ii) Let $F^* = F^*_{X_1}$ be the real claim size distribution. Suppose that the net profit condition (NPC) holds and that $E^*[X_1] < \infty$ (expected value of X_1 with respect to F^*). Further, require that F^* has a probability density and that its integrated tail distribution $F^*_{X_1,I}$ is subexponential.

Calculate a (rough) estimate of the ruin probability $\Psi(u)$ for the safety loading $\rho = 35\%$ and the initial capital $u = 200$ (in millions of Danish Krone) by approximating F^* by means of the empirical distribution function with respect to the fire insurance data above.

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving on day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^{n} X_i^{(m)}$ arriving on day m.

Problem 2. (10 points)

Assume the heterogeneity model for the i-th policy of an insurance. Given $\theta \sim \Gamma(\gamma, \beta)$

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(Gamma distribution) the claim sizes $X_1, ..., X_n$ in the policy are Pareto distributed with parameters (λ, θ), that is

$$P(X_i > x \,|\theta) = (\lambda/x)^\theta, x > \lambda.$$

A reinsurance company only considers the values X_i which exceed a known high treshold K. They "observe" the counting variables $Y_i := 1_{(K,\infty)}(X_i)$ for a known $K > \lambda$. The company is interested in finding an estimate of $p(\theta) := P(X_1 > K \,|\theta)$.

Calculate the linear Bayes estimator and its risk with respect to $p(\theta) := E[Y_1 \,|\theta]$ based on the observations $Y_1, ..., Y_n$, when $K = 55$, $\lambda = 1.5$, $X_1 = 22$, $X_5 = 80$, $X_{10} = 71$ and $X_j = 2$ (in 1000 NOK) for $j = 2, 3, 4, 6, 7, 8, 9$ (in years). Further, suppose that $\beta = \gamma = 2$.

Hint: Recall that the Gamma distribution has the probability density

$$f_{\gamma,\beta}(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, x > 0,$$

where $\Gamma(z)$ is the Gamma function.

Problem 3. (10 points)

We consider again the heterogeneity model and require that in the i-th policy of an insurance the claim numbers $X_t, t \geq 1$ given the heterogeneity parameter θ are Poisson distributed with intensity θ.

(i) Suppose that θ is Gamma distributed with parameters $\gamma, \beta > 0$.
Calculate the linear Bayes estimator $\widehat{\mu}_{LB}$ of the expected claim number $\mu(\theta) := E[X \,|\theta]$ and its corresponding risk for the claim number data $X_1 = 2$, $X_4 = 1, X_5 = 1$, $X_7 = 7, X_{10} = 1$ and $X_j = 0$ for $j = 2, 3, 6, 8, 9$ (in years) and the parameters $\gamma = 1.75, \beta = 1.3$.
(ii) Assume now that $\theta = \exp(Z)$, where $Z \sim Exp(\lambda)$ (exponential distribution) for $\lambda = 3$. Compute $\widehat{\mu}_{LB}$ for the same data as in (i).

Hint: $E[\theta] = \frac{\gamma}{\beta}$, $Var[\theta] = \frac{\gamma}{\beta^2}$, if $\theta \sim \Gamma(\gamma, \beta)$.

Problem 4. (10 points)

Assume Mack's model. The following *run-off triangle* in the figure shows the cumulative paid claims:

Year i	$S_{i,i}$	$S_{i,i+1}$	$S_{i,i+2}$	$S_{i,i+3}$	$S_{i,i+4}$
2015	66238	87434	113473	135235	147162
2016	70233	99167	117432	149111	
2017	89256	123672	150156		
2018	101389	145157			
2019	142381				

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(i) Calculate the predictors $\widehat{S}_{i,m+k}$ of the total claim amounts

$$S_{2016,m+1},$$
$$S_{2017,m+1}, S_{2017,m+2},$$
$$S_{2018,m+1}, S_{2018,m+2}, S_{2018,m+3},$$
$$S_{2019,m+1}, S_{2019,m+2}, S_{2019,m+3}, S_{2019,m+4}.$$

for $m = 2019$ based on the data of the run-off triangle.
(ii) Compute the technical provisions for the year 2021, that is

$$\sum_{j=0}^{2}(\widehat{S}_{2017+j,2021} - \widehat{S}_{2017+j,2020})$$

and for the year 2023, that is

$$\widehat{S}_{2019,2023} - \widehat{S}_{2019,2022}.$$

Hint: Recall that

$$\widehat{S}_{i,m+k} = \widehat{f}^{(m)}_{m-i+k-1} \cdot ... \cdot \widehat{f}^{(m)}_{m-i} \cdot S_{i,m},$$

where

$$\widehat{f}^{(m)}_j = \frac{\sum_{i=1}^{m-j-1} S_{i,i+j+1}}{\sum_{i=1}^{m-j-1} S_{i,i+j}}$$

is the chain ladder estimator of f_j.

Problem 5. (10 points)

Consider a reinsurance company with risk process $U(t) = u + ct - S(t)$, where the total claim amount $S(t)$ is given by

$$S(t) = \sum_{i=1}^{N(t)} (X_i - x)_+.$$

The latter amount corresponds to a so-called *excess-of-loss treaty*. Here $(a)_+ \overset{\text{def}}{=} \max(a, 0)$ for $a \in \mathbb{R}$. Assume that $N(t), t \geq 0$ is a homogeneous Poisson process with intensity $\lambda > 0$, which is independent of the *i.i.d.* claim size sequence $X_i, i \geq 1$. Suppose that $X_1 \sim Exp(\gamma)$ for $\gamma \geq 1$ and that the premium rate c is defined as

$$c = (1 + \rho)\lambda E\left[(X_1 - x)_+\right]$$

for some safety loading $\rho > 0$.

Derive a formula for the ruin probability $\Psi(u) := P(\inf_{t\geq 0} U(t) < 0)$ and calculate $\Psi(u)$ for $u = 20$, $\gamma = \lambda = 1$, $x = 15$ and $\rho = 0.2$.

Hint: Let $Z = (Z_1, ..., Z_n)$ be a random vector with non-negative random variables $Z_i, i = 1, ..., n$. Define for $\lambda_1, ..., \lambda_n \geq 0$

$$\Psi_Z(\lambda_1, ..., \lambda_n) = E\left[\exp(-(\lambda_1 Z_1 + ... + \lambda_n Z_n))\right] \qquad \text{(Laplace-Stieltjes transform)}.$$

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Consider another random vector $Y = (Y_1, ..., Y_n)$ with non-negative random variables $Y_i, i = 1, ..., n$. Use the following fact:
Z and Y have the same probability distribution if and only if $\Psi_Z(\lambda_1, ..., \lambda_n) = \Psi_Y(\lambda_1, ..., \lambda_n)$ for all $\lambda_1, ..., \lambda_n \geq 0$.

End