UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4540 — Non-Life Insurance Mathematics.

Day of examination: Monday, December 3, 2012.

Examination hours: 14:30 – 18:30.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Results you may need
If X and Y are random variables, then

$$E(Y) = E\{\xi(X)\} \qquad \text{and} \qquad \text{var}(Y) = \text{var}\{\xi(X)\} + E\{\sigma^2(X)\}$$

where $\xi(x) = E(Y|X = x)$ and $\sigma^2(x) = \text{var}(Y|X = x)$.

The Poisson density function for N is

$$p(n) = \frac{\lambda^n}{n!}e^{-\lambda}, \quad n = 0, 1, 2, \ldots$$

with mean and standard deviation $E(N) = \lambda$ and $\text{sd}(N) = \sqrt{\lambda}$. If $\mathcal{X} = Z_1 + \cdots + Z_{\mathcal{N}}$ with $Z_1, Z_2, \ldots$ independent and identically distributed and also independent of N, then

$$E(\mathcal{X}) = \lambda\xi_z \qquad \text{and} \qquad \text{var}(\mathcal{X}) = \lambda(\xi_z^2 + \sigma_z^2)$$

where $\xi_z = E(Z_i)$ and $\sigma_z = \text{sd}(Z_i)$.

The Pareto and Gamma density functions for Z are

$$f(z) = \frac{\alpha/\beta}{(1 + z/\beta)^{1+\alpha}}, \quad z > 0 \qquad \text{and} \qquad f(z) = \frac{\alpha^\alpha}{\Gamma(\alpha)\xi_z^{\alpha-1}}z^{\alpha-1}e^{-\alpha z/\xi_z}, \quad z > 0$$

with mean and standard deviation $\xi_z = \beta/(\alpha - 1)$ and $\text{sd}(Z) = \xi_z\sqrt{\alpha/(\alpha - 2)}$ for Pareto and $E(Z) = \xi_z$ and $\text{sd}(Z) = \xi_z/\sqrt{\alpha}$ for Gamma.

Use all these results freely; don't prove them.

Problem 1

a) What is the pure premium of a policy holder? Offer a mathematical definition and explain the actuarial meaning of the concept and its relationship to market premia (at most ten lines!).

(Continued on page 2.)

b) Use a precise mathematical argument and argue that portfolio uncertainty under a suitable condition disappears as the number of policies $J \to \infty$.

c) Extend the argument in b) and analyse a situation where the risk of all policy holders depend on a common random claim intensity μ.

d) What is the reserve of a portfolio? Again offer a mathematical definition and sketch a computer program showing how reserves can be computed when all customers have the same claim intensity μ which may be both fixed and random.

The table shows approximate ϵ-percentiles of the sum of claims during one year for a Poisson/Pareto portfolio. There were $J = 1000$ policies and the parameters of the Pareto distribution were $\alpha = 3$ and $\beta = 2$. Claim numbers assumed common intensity for all policy holders, either a fixed one 0.05 per policy per year or a Gamma-distributed one with mean 0.05 and shape $\alpha = 4$.

ϵ	0.01	0.05	0.25	0.50	0.75	0.95	0.99
	6.5	13.2	29.1	44.9	65.4	104.3	138.2
	25.1	30.6	40.4	48.3	57.5	74.1	91.4

e) Which of the rows of the table belongs to which intensity scenario? Explain briefly and propose assessments of the reserve under both scenarios.

Problem 2

Consider a Poisson/Pareto portfolio with fixed claim intensity μ and shape parameter $\alpha > 1$ in the Pareto distribution. There is an annual re-insurance contract. For each incident $Z^{\text{re}} = \max(Z - b, 0)$ is received by the cedent where b is some threshold.

a) Show that the pure premium of the re-insurance is

$$\pi^{\text{re}} = J\mu \int_b^\infty \{1 - F(z)\}\, dz$$

where $F(z)$ is the distribution function of a claim Z against the cedent.

b) Use this to find an expression for π^{re} in terms of the parameters of the Pareto distribution.

c) Express cedent net responsibility in mathematical terms and explain how you modify the program in Problem 1d) so that it returns cedent net reserve under re-insurance contracts.

d) Argue that if the re-insurance market requires 50% more than the pure premium, then this amount is for cedents an average loss due to re-insurance.

The table shows for the portfolio in Problem 1e) the pure re-insurance premium and the net 99% reserve for the cedent as the threshold b is varied.

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b	infinite	30	20	10	5	4	3	2	1	0
π^{re}	0	0.2	0.4	1.4	4.1	5.6	8.0	12.5	22.0	50.0
Reserve	91.7	87.1	85.5	79.9	71.7	68.2	63.4	55.1	39.4	0

The cedent takes the view that 10% of the capital saved by taking out re-insurance can be regarded as an annual profit for the company.

e) Determine from the table the threshold that is for the cedent optimal when the re-insurance market is as in d).

Suppose the contract is agreed in an inflationary economy with I the rate of inflation per year.

f) How is the fact that claims tend to be more costly a year from now expressed into the Pareto model?

g) Argue that premia will be too low if future inflation is ignored and that this bias will increase with α so that the lighter the tail of the claim distribution the more wrong pricing will become.

Problem 3

This problem examines optimal credibility estimation. Let N be the number of claims from a policy holder during a year and suppose $N|\mu \sim \text{Poisson}(\mu)$ with μ Gamma-distributed with mean ξ_μ and shape α. There is a historical record $N_1, \ldots N_K$ of claim numbers from this customer, and it is then reasonable to assume (as is done in the following) that $N, N_1, \ldots, N_K$ are independent given μ.

a) If $n_1, \ldots, n_K$ are the actual realizations of $N_1, \ldots, N_K$, show that $\mu|n_1, \ldots, n_K$ is Gamma distributed with mean

$$\hat{\mu} = \xi_\mu \frac{\bar{n} + \alpha/K}{\xi_\mu + \alpha/K}$$

and shape $\alpha + K\bar{n}$ where $\bar{n} = (n_1 + \cdots + n_K)/K$.

b) If ξ_z is the expected claim per event (the same for all customers), argue that $\hat{\pi} = \hat{\mu}\xi_z$ is an estimate of the pure premium π of the individual.

c) Show that $\hat{\pi}$ is unbiased; i.e. show that $E(\hat{\pi} - \pi) = 0$.

d) Argue that $\text{sd}(\mu|n_1 + \ldots, n_K) = \hat{\mu}/\sqrt{\alpha + K\bar{n}}$ and deduce that

$$\text{sd}(\hat{\pi} - \pi) = \xi_\mu \xi_z \sqrt{\frac{1}{K\xi_\mu + \alpha}}.$$

e) Interprete the result in d) by explaining why $\text{sd}(\hat{\pi} - \pi) \to 0$ when $K \to \infty$ and $\alpha \to \infty$ and why it reduces to $\xi_z\sqrt{\xi_\mu/K}$ when $\alpha = 0$.

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