UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK4540 — Non-life Insurance Mathematics

Day of examination: Wednesday December 7, 2016

Examination hours: 14.30 – 18.30

This problem set consists of 0 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: Answers in Norwegian and English are equally welcome. Try to express yourself clearly and succinctedly as this will be given weight when your answer is evaluated. You may need the rules of double expectation and double variance, i.e. if $\xi(x) = E(Y|x)$ and $\sigma^2(x) = \text{var}(Y|x)$, then

$$E(Y) = E\{\xi(X)\} \qquad \text{and} \qquad \text{var}(Y) = E\{\sigma^2(X)\} + \text{var}\{\xi(X)\}.$$

Problem 1: Pricing

a) What is a pure premium? Offer a mathematical definition and explain the concept of loading which is a link to the market premium.

Let μ and ξ_z be the mean number of claims annually and the mean claim size for a policy holder with explanatory variables $x_1, \dots, x_v$ and suppose log-linear regressions have been fitted so that we have determined relationships

$$\log(\mu) = \beta_0 + \beta_1 x_1 + \cdots + \beta_v x_v \qquad \text{and} \qquad \log(\xi_z) = \theta_0 + \theta_1 x_1 + \cdots + \theta_v x_v$$

where $\beta_0, \dots \beta_v$ and $\theta_0, \dots, \theta_v$ are coeffcients.

b) How is the pure premium of the policy holder estimated? Is the estimate unbiased? What premium is offered the market if the loading does not depend on $x_1, \dots, x_v$?

Consider a simplified study from motor insurance. There were two explanatory variables which were the age of the policy holder and the car type. Age is divided into "young" (under 25 years) and "old" (25 or more) whereas car type is either a "fast" one (belonging to the GTI-type) and the rest (called "slow"). This categorization amounts to two categories for each of the variables so that $v = 4$ in the regression equations above. Historical data (based on rather limited experience) has lead to the following table:

(Continued on page 2.)

		Age		Car type	
	Constant	"Young"	"Old"	"Slow"	"Fast"
Claim number	-2.90(0.01)	0	-0.47(0.05)	0	0.27(0.07)
Claim size	11.41(1.21)	0	-0.03(0.04)	0	0.18(0.05)

where the numbers in parentheses are the standard deviations of the estimates.

c) Interprete the table when it is to be used for differentiated pricing and propose a rule based on statistically significant explanatory variables.

d) Write down a table where the pure premium is broken down on age and car type.

Problem 2: Control

a) What is meant by the reserve of an insurance portfolio at level $1-\epsilon$? Offer a mathematical definition. Reserve is the same as solvency capital.

Suppose the portfolio is so large that the sum of all payments $\mathcal{X}$ to the policy holders for their claims during one year is Gaussian. Mean and variance are then

$$E(\mathcal{X}) = J\mu\xi_z \qquad \text{and} \qquad \text{var}(\mathcal{X}) = J\mu(\xi_z^2 + \sigma_z^2)$$

where J is the number of policies, μ claim intensity per year and ξ_z and σ_z mean and standard deviation for each claim.

b) Write down a mathematical expression for the reserve when ϕ_ϵ is the $1-\epsilon$ percentile of the standard normal distribution.

There are many situations where μ varies randomly from one year to another so that ξ_μ now is its expectation and σ_μ an additional parameter for its standard deviation.

c) Show that the mean and standard deviation of $\mathcal{X}$ now becomes

$$E(\mathcal{X}) = J\xi_\mu\xi_z \qquad \text{and} \qquad \text{var}(\mathcal{X}) = J\xi_\mu(\xi_z^2 + \sigma_z^2) + J^2\xi_z^2\sigma_\mu^2.$$

and argue that the expressions collapse to the former ones when μ is fixed.

d) Under what condition is this portfolio perfectly diversified with uncertainty losing all practical importance as the number of policies become large? Explain what happens in general.

One possible specification of μ is $\mu = \xi_\mu G_\alpha$ where G_α is a Gamma variable with mean 1 and shape α. Assume this model in the following.

e) Sketch a program simulating $\mathcal{X}$ when you assume Gaussian and Gamma samplers to be available and explain how the simulations can be used to compute the reserve.

Let $J = 100000$, $\xi_\mu = 5\%$, $\xi_z = 1$ and $\sigma_z = 0.25$. The 99% reserve for fixed $\mu = \xi_\mu$ is the one used in practice, but we now also consider random μ with $\alpha = 25$ (implying $\text{sd}(\mu) = 1\%$).

(Continued on page 3.)

f) The two reserve evaluations are approximately 50650 and 76360, but which one belongs to which model for μ? Comment on the error in the version for fixed μ.

Problem 3: Solvency II
Note: Problems c), d) and e) can be answered independently of whether or not you have completed a) and b).

The Solvency II system of regulations is in force in The European Union and the EØS countries from January 2016, and all insurance companies have to abide by it. We shall below evaluate the so-called non-life solvency capital requirement $\text{SCR}_{\text{non-life}}$ for an extremely simplified company where there are three portfolios (or Lines of Business known as **LOB**). The first one (of the so-called **reserve** risk type) has outstanding claims extending over three years that are evaluated by the chain ladder method, the second one (**premium** risk) has no outstanding claims at all, but there will be new business during the coming year and the third one is catastrophe insurance, again with no claims pending at the start of the year. Solvency II specifies

$$\text{SCR}_{\text{non-life}} = \left((\text{SCR}_{\text{preres}})^2 + (\text{SCR}_{\text{cat}})^2 + 1.5 \times \text{SCR}_{\text{preres}} \times \text{SCR}_{\text{cat}}\right)^{1/2}$$

where $\text{SCR}_{\text{preres}}$ is the solvency capital requirement for the first two LOB's merged together and SCR_{cat} the same for the third one.

The following table represents the history of the first LOB at the end of 2015. The numbers shown are the cumulative claims settled after k = 0, 1, 2 and 3 years with money unit 1000 NOK. Everything after the year 2015 represents the future and is not known.

	Development year k			
	$k=0$	$k=1$	$k=2$	$k=3$
2012	100	200	250	275
2013	200	380	430	
2014	250	450		
2015	200			

a) Demonstrate that the aggregation factors of the chain ladder method become $\hat{f}_1 = 1.87$, $\hat{f}_2 = 1.17$ and $\hat{f}_3 = 1.10$.

b) Fill in the empty entries in the table above and write down estimates for the amounts the company has to pay in 2016, 2017 and 2018.

With suitable discounting the present value of these payments at the start of 2016 becomes CL= 2731 (in Solvency II notation) which you will need below. You will also need the expected premium P= 2000 for the second LOB for the coming year and the Solvency II standard deviation factors $\sigma_1 = 0.1$ and $\sigma_2 = 0.1$ for the first two LOB's. The Solvency II SCR for the premium/reserve component may then become

$$\text{SCR}_{\text{preres}} = \sqrt{\sigma_1^2 + \sigma_2^2} \times (\text{CL} + \text{P}) \times q \qquad \text{where} \qquad q = 3.$$

c) Try to motivate this expression by explaining what the three factors represent and compute $\text{SCR}_{\text{preres}}$ numerically.

(Continued on page 4.)

The third LOB deals with catastrophe insurance, and here the company has been allowed by the authorities to develope a so-called internal model rather than following the Solvency II specifications. Details are as follows: With X the value of all damages caused by an event, the insurance only covers up to some maximum S so that the responsibility of the company becomes $X_{\text{cat}} = \min(X, S)$, but this is gross. The company has also bought reinsurance and will receive from the reinsurer the amount $X_{\text{re}} = \min(X_{\text{cat}} - a, 0)$ where $a < S$ is specified by the contract. It is assumed that X follows a Pareto distribution with density function

$$f(x) = \frac{\alpha/\beta}{(1 + x/\beta)^{1+\alpha}}$$

where α and β are parameters.

d) Argue that the net catastrophe responsibility is $X_{\text{ce}} = \min(X, a)$ and deduce that the 99.5% solvency capital for cat risk is

$$\text{SCR}_{\text{cat}} = \min\{\beta(0.005^{-1/\alpha} - 1), a\}.$$

e) If $\beta = 2000$, $\alpha = 3$ and $S = 10000$, compute SCR_{cat} when $a = S$ (no reinsurance), $a = 5000$ and $a = 0$ (reinsurer takes all risk) and also $\text{SCR}_{\text{non-life}}$. Comment on the significance of buying reinsurance.

END

(Continued on page 5.)