UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Candidate no

Constituent exam in: STK4540 — Non-life Insurance Mathematics

Day of examination: 12th December 2018

Examination hours: 09:00 – 13:00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This grade chart is only a guideline:

A (85-100) , B (70-85), C (58-70), D (49-57) , E (40-48), F (0-39).

The final grade is set based on an evaluation of the exam as a whole.

Problem 1

(a) The Cramér-Lundberg model is a model for the total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$ where N is specified to be a homogeneous Poisson process, $\{X_i\}_{i\geq 1}$ are i.i.d. and independent of N. More concretely, we have the following specifications:

- Claims happen at the arrival times $0 \leq T_1 \leq T_2 \leq \dots$ of a homogeneous Poisson process $N(t) = \#\{i \geq 1 : T_i \leq t\}, t \geq 0$.
- The ith claim arriving at time T_i causes the claim size X_i. The sequence $\{X_i\}_{i\geq 1}$ constitutes an i.i.d. sequence of non-negative random variables.
- The sequences $\{T_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ are independent. In particular, N and $\{X_i\}_{i\geq 1}$ are independent.

Give 1p if the answer is correct. Give 0.5p if N is chosen to be a homogeneous Poisson process.

(b) The risk process is defined as

$$U(t) = u + p(t) - S(t), \quad t \geq 0,$$

i.e. initial capital (usually big), plus income from the premiums minus the claims. The event of ruin is the event

$$R = \{\omega : U(t,\omega) < 0 \text{ for some } t \geq 0\}.$$

(Continued on page 2.)

The probability of ruin is then $P(R)$, which has several equivalent definitions

$$P(R) = P(\inf\{t > 0 : U(t) < 0\} < \infty).$$

Give 1p if the three definitions are correct. Give 0.5p if risk process is correct, and 0.25p for ruin and probability of ruin.

(c) Let, in the renewal model, $W_i = T_i - T_{i-1}$, $i \geq 1$, $T_0 := 0$ denote the interarrival times. Let $Z_i := X_i - cW_i$, $i \geq 1$. Since X_i and W_i are i.i.d. we can look at $Z_1 = X_1 - cW_1$. A necessary condition to hope for ruin with probability different from one is the so-called Net Profit Condition (NPC) and it is given by

$$E[Z_1] = E[X_1] - cE[W_1] < 0,$$

which implies

$$c > \frac{E[X_1]}{E[W_1]}.$$

If in addition, we are in the Cramér-Lundberg model then $W_1 \sim Exp(\lambda)$ being λ the intensity of the Poisson process N and the NPC is $c > \lambda E[X_1]$. The condition is rather intuitive; $E[W_1]$ represents the average time between claims, which is exactly when we obtain the linear income $cE[W_1]$ (in average). This amount should be bigger than the average claim size generated at time T_1.

Give 1p for the statement of the NPC, give 0.5p if the candidate states the condition in the Cramér-Lundberg model, and give special consideration to a student who has explained the interpretation of NPC.

(d) Choosing $f(x) = e^{hx}$ we immediately have

$$P(X \geq a) \leq \frac{m_X(h)}{e^{ha}} = e^{-ha} m_X(h),$$

which shows that the tail $P(X \geq a)$ has exponential decay. The existence of the moment generating function of a claim size around the origin implies exponential decay, which in particular, is a bad model if the want to model *large claim sizes* which is a common feature in non-life insurance.

Give 0.5p for the proof and 0.5p for the explanation related to small/large claims

Problem 2

(a) First observe that the distribution of the X_i's is given by

$$P(X_i = 1) = P(\omega_1) = 0.2, P(X_i = 2) = P(\omega_2) = 0.7, P(X_i = 3) = P(\omega_3) = 0.1.$$

(Continued on page 3.)

Fix $t \geq 0$. Since $P(X_1 = 0) = 0$ we have $p_0(t) = P(N(t) = 0) = e^{-\lambda t}$. Then for $n = 1$ using the formula in the exercise we have

$$p_1(t) = \lambda t P(X_1 = 1) p_0(t) = 0.2\lambda t e^{-\lambda t}.$$

For $n = 2$:

$$\begin{aligned} p_2(t) &= \frac{\lambda t}{2}\left(P(X_1 = 1)p_1(t) + 2P(X_2 = 2)p_0(t)\right) \\ &= \frac{\lambda t}{2}\left(0.2 \cdot 0.2\lambda t e^{-\lambda t} + 2 \cdot 0.7 e^{-\lambda t}\right) \\ &= 0.2\lambda^2 t^2 e^{-\lambda t} + 0.7\lambda t e^{-\lambda t}. \end{aligned}$$

For $n \geq 3$ we then have

$$p_n(t) = \frac{\lambda t}{n}\left(0.2 p_{n-1}(t) + 1.4 p_{n-2}(t) + 0.3 p_{n-3}(t)\right).$$

Give 0.25p for $p_0(t)$, $p_1(t)$ and $p_2(t)$. Give 1p for everything. Do not penalize for not simplifying the expressions.

(b) We will need $E[X_1]$ and $Var[X_1]$:

$$E[X_1] = 0.2 + 2 \cdot 0.7 + 3 \cdot 0.1 = 0.2 + 1.4 + 0.3 = 1.9$$

and

$$E[X_1^2] = 0.2 + 2^2 \cdot 0.7 + 3^2 \cdot 0.1 = 0.2 + 2.8 + 0.9 = 3.9$$

Hence,

$$Var[X_1] = E[X_1^2] - E[X_1]^2 = 3.9 - 1.9^2 = 0.29$$

By the independence of $\{X_i\}_{i \geq 1}$ and N and a conditioning argument we know that

$$E[S(t)] = E[X_1]E[N(t)] = \lambda t E[X_1] = 1.9\lambda t.$$

By the law of total variance we have

$$\begin{aligned} Var[S(t)] &= Var[E[S(t)|N(t)]] + E[Var[S(t)|N(t)]] \\ &= Var[N(t)E[X_1]] + E[N(t)Var[X_1]] \\ &= E[X_1]^2 Var[N(t)] + E[N(t)]Var[X_1] \\ &= 1.9^2\lambda t + \lambda t 0.29 \\ &= 3.9\lambda t. \end{aligned}$$

The Central Limit Theorem says that for large t we have

$$\frac{S(t) - 1.9\lambda t}{\sqrt{3.9\lambda t}} \approx N(0, 1).$$

Give 0.25p for computing $E[X_1]$ and $Var[X_1]$. Give 0.5p for computing $E[S(t)]$ and $Var[S(t)]$. Give 1p if in addition the CLT is stated.

(Continued on page 4.)

(c) By the first year we will have observed the random number of claims $N(1)$. Hence, $X_1, \dots, X_{N(1)}$. Observing no thefts means that there is no X_j such that $X_j = 1$ for all $j = 1, \dots, N(1)$. Hence, if we call $\mathcal{A}$ the event that no policyholders report a theft during the first year then

$$P(\mathcal{A}) = P(X_1 \neq 1, \dots, X_{N(1)} \neq 1).$$

Using conditional probability and the law of total probability we have

$$P(\mathcal{A}) = \sum_{n=0}^{\infty} P(X_1 \neq 1, \dots, X_n \neq 1 | N(1) = n) P(N(1) = n).$$

Since all X_i are independent and equally distributed we have

$$P(\mathcal{A}) = \sum_{n=0}^{\infty} P(X_1 \neq 1)^n \frac{e^{-\lambda}\lambda^n}{n!} = e^{\lambda(P(X_1 \neq 1) - 1)} = e^{10(0.8-1)} \approx 0.13534.$$

Give 0.25p if the candidate tried something meaningful. Otherwise give 1p for the exercise.

(d) We are under the Cramér-Lundberg model so we need to charge c such that

$$c > \lambda E[X_1] = 10\,(1000 \cdot 0,2 + 2000 \cdot 0,7 + 3000 \cdot 0,1) = 19\,000 NOK.$$

Give 0.25p if the candidate understands that one has to use the NPC. Otherwise 1p for everything.

Problem 3

(a) We do not know about the law of X_i and hence about $\overline{X}_n$. But we know that conditionally on θ, the X_i's are i.i.d. and Weibull distributed. Using the law of total variance we have

$$Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = Var\left[E\left[\frac{1}{n}\sum_{i=1}^{n} X_i | \theta\right]\right] + E\left[Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i | \theta\right]\right].$$

Hence,

$$\begin{aligned}
Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} E\left[X_i|\theta\right]\right] + E\left[\frac{1}{n^2}\sum_{i=1}^{n} Var\left[X_i|\theta\right]\right] \\
&= Var\left[E\left[X_1|\theta\right]\right] + E\left[\frac{1}{n} Var\left[X_1|\theta\right]\right] \\
&= Var\left[\theta^{1/\lambda}\Gamma\left(1+1/\lambda\right)\right] + \frac{1}{n} E\left[\theta^{2/\lambda}\left(\Gamma\left(1+2/\lambda\right) - \Gamma\left(1+1/\lambda\right)^2\right)\right] \\
&= \Gamma\left(1+1/\lambda\right)^2 Var[\theta^{1/\lambda}] + \frac{\Gamma\left(1+2/\lambda\right) - \Gamma\left(1+1/\lambda\right)^2}{n} E[\theta^{2/\lambda}].
\end{aligned}$$

(Continued on page 5.)

We can observe that the variance of this estimator does not vanish as n tends to infinity unless θ is deterministic.

The computation of the variance does not need to be very specific. It is enough to conclude that it does not tend to 0. Give 1p. Give something if there is a minimum effort to justify why it is a bad estimator.

(b) The posterior density of θ given $\vec{X} = \vec{x}$ is

$$f_\theta(y|\vec{X} = \vec{x}) \propto \left(\prod_{i=1}^{n} f_{X_1}(x_i|\theta = y) \right) f_\theta(y).$$

We only care about factors depending on y. Hence,

$$f_\theta(y|\vec{X} = \vec{x}) \propto \left(\prod_{i=1}^{n} \frac{\lambda}{y} x_i^{\lambda-1} e^{-\frac{x_i^\lambda}{y}} \right) \left(\frac{1}{y} \right)^{\alpha+1} e^{-\frac{\beta}{y}}$$
$$\propto \left(\frac{1}{y} \right)^{n} \left(\frac{1}{y} \right)^{\alpha+1} e^{-\sum_{i=1}^n x_i^\lambda \frac{1}{y}} e^{-\beta \frac{1}{y}}$$
$$= \left(\frac{1}{y} \right)^{n+\alpha+1} e^{-\frac{\sum_{i=1}^n x_i^\lambda + \beta}{y}}$$

which gives rise to an inverse Gamma density with parameter $\overline{\alpha} = n + \alpha$ and $\overline{\beta} = \sum_{i=1}^n x_i^\lambda + \beta$.

Give 0.25p if the student writes the relation between posterior, likelihood and prior.

(c) The net premium is given by

$$\mu(\theta) = E[X_1|\theta] = \theta^{1/\lambda} \Gamma(1 + 1/\lambda).$$

The Bayes estimator $\widehat{\mu}_B$ is the posterior mean of the random quantity above. Hence,

$$\widehat{\mu}_B = \Gamma(1 + 1/\lambda) E[\theta^{1/\lambda}|\vec{X}].$$

Let us focus on $E[\theta^{1/\lambda}|\vec{X}]$. This is the posterior expectation, hence

$$E[\theta^{1/\lambda}|\vec{X}] = \int_0^\infty y^{1/\lambda} \frac{\overline{\beta}^{\overline{\alpha}}}{\Gamma(\overline{\alpha})} \left(\frac{1}{y} \right)^{\overline{\alpha}+1} e^{-\frac{\overline{\beta}}{y}} dy$$
$$= \frac{\overline{\beta}^{\overline{\alpha}}}{\Gamma(\overline{\alpha})} \int_0^\infty \left(\frac{1}{y} \right)^{n+\alpha-\frac{1}{\lambda}+1} e^{-\frac{\overline{\beta}}{y}} dy$$
$$= \frac{\overline{\beta}^{\overline{\alpha}}}{\Gamma(\overline{\alpha})} \frac{\Gamma\left(n + \alpha - \frac{1}{\lambda}\right)}{\overline{\beta}^{n+\alpha-\frac{1}{\lambda}}}$$
$$= \frac{\Gamma\left(n + \alpha - \frac{1}{\lambda}\right)}{\Gamma(n + \alpha) \left(\sum_{i=1}^n X_i^\lambda + \beta\right)^{-1/\lambda}}.$$

(Continued on page 6.)

As a result, the Bayes estimator is

$$\widehat{\mu}_B = \frac{\Gamma(1+1/\lambda)\,\Gamma\left(n+\alpha-\frac{1}{\lambda}\right)}{\Gamma(n+\alpha)}\left(\sum_{i=1}^{n} X_i^\lambda + \beta\right)^{1/\lambda}.$$

Give 0.1p if the student has computed $\mu(\theta)$. Give 0.25p if the candidate has said that the Bayes estimator is the posterior mean of $\mu(\theta)$. Give 0.5p if there are meaningful computations towards the Bayes estimator. Give 1p for everything

(d) Take $\widehat{\mu}_B$ and express the estimate as follows

$$\widehat{\mu}_B = \frac{\Gamma(1+1/\lambda)\,\Gamma\left(n+\alpha-\frac{1}{\lambda}\right)n^{1/\lambda}}{\Gamma(n+\alpha)}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^\lambda + \frac{\beta}{n}\right)^{1/\lambda}.$$

Define $Y_i := X_i^\lambda$, then $\{Y_i\}_{i=1}^n$ are conditionally on θ i.i.d. and the strong law of large numbers implies

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow[n\to\infty]{a.s.} E[Y_1|\theta] = E[X_1^\lambda|\theta].$$

Furthermore,

$$\lim_n \frac{\Gamma\left(n+\alpha-\frac{1}{\lambda}\right)n^{1/\lambda}}{\Gamma(n+\alpha)} = 1.$$

Hence,

$$\widehat{\mu}_B \xrightarrow[n\to\infty]{a.s.} \Gamma(1+1/\lambda)\,E[X_1^\lambda|\theta]^{1/\lambda}.$$

Finally, we need to show that $E[X_1^\lambda|\theta] = \theta$. Indeed,

$$\begin{aligned} E[X_1^\lambda|\theta] &= \int_0^\infty y^\lambda \frac{\lambda}{\theta} y^{\lambda-1} e^{-\frac{y^\lambda}{\theta}}\,dy \\ &= \int_0^\infty \frac{\lambda}{\theta} y^{2\lambda-1} e^{-\frac{y^\lambda}{\theta}}\,dy \\ &= \frac{\lambda}{\theta}(\theta^{1/\lambda})^{2\lambda}\frac{\Gamma\left(\frac{2\lambda}{\lambda}\right)}{\lambda} \\ &= \theta. \end{aligned}$$

Give 0.5p if the candidate applied the limit and the strong law of large numbers to obtain a limiting expression but does not compute the conditional expectation. Give 1p for everything.

(Continued on page 7.)

Problem 4

Using the formula for the CLM estimate given in the exercise we can fill in the run-off triangle and obtain

Cumulative claims loss settlements	Development year				
Claims occurrence year	0	1	2	3	4
2011	1143	1647	2165	2382	2673
2012	1387	1894	2384	2745	3080
2013	1656	2165	2763	3114	3495
2014	1957	2294	2940	3313	3718
2015	2195	2859	3663	4129	4633
CLM estimator for claims loss settlement factor		1,3023	1,2815	1,1271	1,1222

Figure 1: Observed and estimated cumulative payments.

The technical provisions for year, say 2016, are the sum of the incremental claims loss settlements. This corresponds to the incremental settlements placed in the diagonal corresponding to year 2016, that is $(3080 - 2745) + (3114 - 2763) + (2940 - 2294) + (2859 - 2195) = 1996$. Repeating this procedure for each year (increments in each diagonal) we obtain the following table:

Calendar year	Estimated claims loss settlement amounts
2016	1996
2017	1559
2018	870
2019	504

Figure 2: Technical provisions for future years

The provisions for the years 2017 and 2019 are 1559 and 504, respectively.

Give 1p if the candidate has computed the estimated claims loss settlements. Give 1p for the estimated loss settlement amounts for years 2017 and 2019.

The final point sum is a number X between 0 and 14. The grade is then computed as a number between 0 and 100 as follows

$$\text{Grade} = \frac{100}{14} X.$$