Solutions to Ex # 1

1.
$$E(s^2) = \frac{1}{n-1} E\Big\{ \sum_{i\in s} (y_i - \bar y_s)^2 \Big\}$$

$$= \frac{1}{n-1} E\Big\{ \sum_{i\in s} (y_i - \mu + \mu - \bar y_s)^2 \Big\}$$

$$= \frac{1}{n-1} E\Big\{ \sum_{i\in s} (y_i-\mu)^2 + n(\bar y_s - \mu)^2 + 2(\mu - \bar y_s) \overbrace{\sum_{i\in s}(y_i-\mu)}^{n(\bar y_s-\mu)} \Big\}$$

$$= \frac{1}{n-1} E\Big\{ \sum_{i\in s} (y_i-\mu)^2 - n(\bar y_s-\mu)^2 \Big\}$$

$$= \frac{1}{n-1} \Big\{ E\sum_{i=1}^{N} (y_i-\mu)^2 Z_i - n\,\mathrm{Var}(\bar y_s) \Big\}$$

$$= \frac{1}{n-1} \Big\{ \frac{n}{N}\cdot (N-1)\sigma^2 - \Big(1-\frac{n}{N}\Big)\sigma^2 \Big\}$$

$$= \frac{1}{n-1} \Big\{ n\sigma^2 - \frac{n}{N}\sigma^2 + \frac{n}{N}\sigma^2 - \sigma^2 \Big\} = \frac{(n-1)\sigma^2}{n-1} = \sigma^2$$

Alternative derivation, directly:

$$E\Big\{ \sum_{i\in s} (y_i - \bar y_s)^2 \Big\} = E\sum_{i\in s} y_i^2 - n E(\bar y_s^2)$$

1) $n E(\bar y_s^2) = n[\mathrm{Var}(\bar y_s) + \mu^2] = \sigma^2\Big(1-\frac{n}{N}\Big) + n\mu^2$

2) $E\sum_{i\in s} y_i^2 = E\sum_{1}^{N} y_i^2 Z_i = \frac{n}{N}\sum y_i^2 = \frac{n}{N}\sum (y_i-\mu)^2 + n\mu^2$

$$\Rightarrow E\Big\{ \sum_{i\in s} (y_i - \bar y_s)^2 \Big\} = \frac{n}{N}(N-1)\sigma^2 - \sigma^2\frac{N-n}{N}$$

$$= \frac{\sigma^2}{N}\{nN - n - N + n\} = (n-1)\sigma^2.$$

2. $\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i-\mu)^2 \qquad \mu = \frac{t}{N} = p$

$$\sum_{i=1}^{N}(y_i-p)^2 = \sum_{\{i:y_i=1\}}(1-p)^2 + \sum_{\{i:y_i=0\}}p^2$$

$$= Np\cdot(1-p)^2 + (N-Np)p^2$$

since $\#\{i: y_i=1\} = N.p$ and $\#\{i:y_i=0\} = N-Np$

$$\Rightarrow \sum_{i=1}^{N}(y_i-p)^2 = Np(1-p)\{1-p+p\} = Np(1-p).$$

3. $Var(\hat{t}_1) - Var(\hat{t}_2)$

$$= \frac{1}{3}\Big\{\left(\tfrac{1}{2}y_1+y_3-y_2-\tfrac{1}{2}y_3\right)^2 - \left(\tfrac{1}{2}y_1+y_3-y_2\right)^2$$
$$+ \left(\tfrac{1}{2}y_2-y_1+\tfrac{1}{2}y_3\right)^2 - \left(\tfrac{1}{2}y_2-y_1\right)^2\Big\}$$

$$= \frac{1}{3}\Big\{ -y_3\left(\tfrac{1}{2}y_1+y_3-y_2\right)+\tfrac{1}{4}y_3^2$$
$$+ y_3\left(\tfrac{1}{2}y_2-y_1\right)+\tfrac{1}{4}y_3^2\Big\}$$

$$= \frac{1}{3}y_3\left\{\tfrac{1}{2}y_3 - \tfrac{1}{2}y_1 - y_3 + y_2 + \tfrac{1}{2}y_2 - y_1\right\}$$

$$= \frac{1}{3}y_3\left\{-\tfrac{1}{2}y_3 - \tfrac{3}{2}y_1 + \tfrac{3}{2}y_2\right\}$$

$$= \frac{1}{6}y_3\left(3y_2 - 3y_1 - y_3\right)$$