UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-BIO2100 — Mathematical Biology

Day of examination: Tuesday June 3 2014

Examination hours: 09.00 – 13.00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: All handwritten and printed aids in addition to approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider an insect population where adults lay eggs in the spring and then die. The eggs hatch into larvae which eat and grow and then overwinter in a pupal stage. The adults emerge from the pupae in the following spring. We count the adults in the breeding seasons. Denote by N_k the number of adults in the kth breeding season. The population size is assumed to follow the Beverton-Holt model:

$$N_{k+1} = \frac{R_0 N_k}{1 + aN_k}, \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where N_0, R_0, a are positive constants; R_0 is the average number of eggs laid by an adult, the so-called basic reproductive ratio.

1a

Classify the Beverton-Holt model according to competition (under-, exact- or overcompensation). Justify your answer.

We refer to a positive number K as the *carrying capacity of the environment* if the population having reached K, will stay there: $N_k = K$ for some $k = 0, 1, 2, \ldots$ implies $N_{k+m} = K$ for all $m = 0, 1, 2, \ldots$.

Given a constant $K > 0$, how do you have to choose $a > 0$ and $R_0 > 0$ to ensure that K becomes the carrying capacity of the environment.

1b

Determine the nonnegative steady state solutions of (1).

1c

Determine the stability of the steady states found in **1b**.

(Continued on page 2.)

1d

Suppose $a = 1$ and $R_0 = 1$. Show that the solution of the Beverton-Holt model (1) in this case is given by

$$x_k = k + x_0, \qquad k = 0, 1, 2, \ldots,$$

where $x_k := \frac{1}{N_k}$. What happens with N_k as $k \to \infty$.

Problem 2

Schistosomiasis is a disease caused by parasitic worms. The parasites live in freshwater snails. The infectious form of the parasite, known as cercariae, emerge from the snail, hence contaminating water. You can become infected when your skin comes in contact with contaminated freshwater. Denote by M the number of worms in human hosts, N the total number of human hosts, and I the number of infected snails. A possible model for the disease is given by the following system of differential equations:

$$\begin{aligned} \frac{dM}{dt} &= \beta_1 N - \delta M, \\ \frac{dI}{dt} &= \beta_2 (N - I) - dI, \end{aligned} \tag{2}$$

where δ, d are positive constants and the functions β_1, β_2 represent the force of infection for the human and snail populations.

2a

Suppose $d = 0$ and take

$$\beta_1 = cI, \qquad \beta_2 = \frac{bM^2}{N(M+N)},$$

where c, b are positive constants. Determine the steady states for (2) and their stability properties.

2b

Consider (2). Suppose that the quasi-steady state approximation $\frac{dI}{dt} \approx 0$ holds. Use this to derive a single differential equation for M.

2c

Assume that the quasi-steady state approximation $\frac{dI}{dt} \approx 0$ holds. Suppose $d = 0$ and $M(t) > \frac{cN^2}{\delta}$ for all finite $t \geq 0$. Use the result from **2b** to explain what happens with $M(t)$ as $t \to \infty$.

Problem 3

Malaria is a vector borne disease that is spread by a small parasite. Mosquitos infect humans, which in turn can infect other mosquitos. The infectious agent is a parasite which is infected into the human blood by a

(Continued on page 3.)

mosquito. The parasite develops in the human and produce germs that can be taken up by a biting mosquito (thereby infecting the mosquito). In what follows, we will consider a SIS model. Denote by (S_1, I_1) the susceptible and infectious human populations, respectively. Similarly, we denote by (S_2, I_2) the susceptible and infectious mosquito populations, respectively. The relevant system of ordinary differential equations is given by

$$\begin{aligned}
\frac{dS_1}{dt} &= -ap_1\frac{S_1}{N_1}I_2 + \gamma_1 I_1,\\
\frac{dI_1}{dt} &= ap_1\frac{S_1}{N_1}I_2 - \gamma_1 I_1,\\
\frac{dS_2}{dt} &= -ap_2 S_2\frac{I_1}{N_1} + \gamma_2 I_2 + c_2N_2 - c_2S_2,\\
\frac{dI_2}{dt} &= ap_2 S_2\frac{I_1}{N_1} - \gamma_2 I_2 - c_2 I_2,
\end{aligned} \tag{3}$$

where $a, p_1, p_2, \gamma_1, \gamma_2, c_2$ are given positive constants, and N_1, N_2 are respectively the human and mosquito population sizes.

3a

Argue why N_1 and N_2 are constants (i.e., independent of time t). Define

$$u_1 = S_1/N_1, \qquad v_1 = I_1/N_1$$

and

$$u_2 = S_2/N_2, \qquad v_2 = I_2/N_2.$$

Show that u_1, v_1, u_2, v_2 satisfy the following system:

$$\begin{aligned}
\frac{du_1}{dt} &= -ap_1\frac{N_2}{N_1}u_1v_2 + \gamma_1 v_1,\\
\frac{dv_1}{dt} &= ap_1\frac{N_2}{N_1}u_1v_2 - \gamma_1 v_1,\\
\frac{du_2}{dt} &= -ap_2u_2v_1 + \gamma_2v_2 + c_2 - c_2u_2,\\
\frac{dv_2}{dt} &= ap_2u_2v_1 - (\gamma_2 + c_2)v_2,
\end{aligned} \tag{4}$$

along with the algebraic constraints $u_1 + v_1 = 1$, $u_2 + v_2 = 1$.

3b

In what follows, we shall focus on the equations for v_1 and v_2. Show that these equations can be written in the following form:

$$\begin{aligned}
\frac{dv_1}{dt} &= \gamma_1(\alpha_1(1 - v_1)v_2 - v_1),\\
\frac{dv_2}{dt} &= (\gamma_2 + c_2)(\alpha_2(1 - v_2)v_1 - v_2),
\end{aligned} \tag{5}$$

where $\alpha_1 = \frac{ap_1N_2}{\gamma_1N_1}$ and $\alpha_2 = \frac{ap_2}{\gamma_2+c_2}$. Determine the steady states for (5). Define the basic reproductive ratio as

$$R_0 := \alpha_1\alpha_2 = \frac{a^2p_1p_2N_2}{\gamma_1N_1(\gamma_2 + c_2)}.$$

Identify a condition on R_0 ensuring the existence of an endemic (non-trivial) steady state.

(Continued on page 4.)

3c

Determine the stability of the steady state solutions.

END