Notes for MAT-INF1310 – 1
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1 Calculus

Exercise 1.1 *Suppose u is a function of class C^1 on some open interval I, which is twice differentiable at a point $t \in I$. Prove :*

$$\ddot{u}(t) = \lim_{h \to 0} \frac{u(t+h) - 2u(t) + u(t-h)}{h^2}. \tag{1}$$

2 Transforming equations

Exercise 2.1 *Transform the initial value problem for the second order scalar equation:*

$$\ddot{\theta}(t) = -\sin\theta(t), \tag{2}$$
$$\theta(0) = 0, \tag{3}$$
$$\dot{\theta}(0) = 1. \tag{4}$$

into an equivalent initial value problem for a first order equation in dimension 2.

Exercise 2.2 *Transform the initial value problem for the non-autonomous scalar equation:*

$$\dot{\theta}(t) = -t\theta(t), \tag{5}$$
$$\theta(0) = 0, \tag{6}$$

into an equivalent initial value problem for an autonomous first order equation in dimension 2.

3 Uniqueness of solutions

Exercise 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:*

$$f(x) = \sqrt{x} \quad \textit{for} \quad x > 0, \tag{7}$$
$$f(x) = 0 \quad \textit{for} \quad x \leq 0. \tag{8}$$

Find three distinct solutions to the initial value problem:

$$\dot{x}(t) = f(x(t)), \quad \textit{for} \quad t \geq 0, \tag{9}$$
$$x(0) = 0, \tag{10}$$

Hint: Find one solution in the form $t \mapsto \alpha t^\beta$.

Notice that in the following exercise we do not assume that f is Lipschitz.

Exercise 3.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous non-increasing function. Pick $x_0 \in \mathbb{R}$. Show that the initial value problem:*

$$\dot{x}(t) = f(x(t)), \quad \text{for} \quad t \geq 0, \tag{11}$$
$$x(0) = x_0, \tag{12}$$

has at most *one solution (for $t \geq 0$).*

The following is a more elaborate version of the preceding exercise:

Exercise 3.3 *Denote by $(x, y) \mapsto x \cdot y$ the (standard) Euclidean scalar product on $\mathbb{R}^n$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that:*

$$\forall x, y \in \mathbb{R}^n \quad (F(x) - F(y)) \cdot (x - y) \leq 0. \tag{13}$$

Show that for any given $x_0 \in \mathbb{R}^n$ the initial value problem:

$$\dot{x} = F(x), \tag{14}$$
$$x(0) = x_0, \tag{15}$$

has at most one solution (for positive times!). Prove that if $F(0) = 0$ then this solution is bounded.

For the following exercise it might be useful to know the definition and existence of a greatest lowerbound $\inf A$ for subsets A of $\mathbb{R}$ which are bounded below[1].

Exercise 3.4 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and that $f(0) = 0$. Suppose that $x : t \mapsto x(t)$ satisfies:*

$$\dot{x} \leq f(x), \tag{16}$$
$$x(0) \leq 0. \tag{17}$$

Show that for all $t \geq 0$, $x(t) \leq 0$.

Hint: Suppose on the contrary that for some $t_1 > 0$, $x(t_1) > 0$. Construct a t_2 such that $t_2 < t_1$ and $x(t) \geq 0$ for all $t \in [t_2, t_1]$, and $x(t_2) = 0$. Obtain a contradiction by applying a Gronwall estimate on $[t_2, t_1]$.

The following exercise is a more elaborate version of the preceding one:

Exercise 3.5 *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and Lipschitz in the sense that for some $M \in \mathbb{R}_+$:*

$$\forall t, x, y \quad |f(t, x) - f(t, y)| \leq M|x - y|. \tag{18}$$

Pick $x_0 \in \mathbb{R}$. Suppose that $x : \mathbb{R}_+ \to \mathbb{R}$ is a (continuously differentiable) function solving the initial value problem:

$$\dot{x}(t) = f(t, x(t)) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x(0) = x_0. \tag{19}$$

Suppose that $y : \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable function satisfying:

$$\dot{y}(t) \leq f(t, y(t)) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad y(0) \leq x_0. \tag{20}$$

Prove that:

$$\forall t \geq 0 \quad y(t) \leq x(t). \tag{21}$$

[1] A lower bound of A is an $x \in \mathbb{R}$ such that $\forall y \in A \quad x \leq y$. Let $L(A)$ denote the set of lower bounds of A. If $L(A)$ is non-empty then $L(A)$ has a greatest element: that is, there is a (unique) element $x \in L(A)$ such that $\forall y \in L(A) \quad y \leq x$. It is denoted $\inf A$.