Notes for MAT-INF1310 – 2
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(the first set of notes was modified and detailed on January 25-th in the evening)

1 Coordinates and vector notations

Let $n \geq 1$ be an integer. In $\mathbb{R}^{n+1}$ the first variable will be called time, whereas the n other will be called space variables. Let $f_i : \mathbb{R}^{n+1} \to \mathbb{R}$ for $i \in \{1, \cdots, n\}$ be n continuous functions. Pick an element $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. We consider the problem of finding an interval $I \subset \mathbb{R}$ containing 0 and n differentiable functions $x_i : I \to \mathbb{R}$ (for $i \in \{1, \cdots, n\}$) satisfying the differential equations:

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(t, x_1(t), \cdots, x_n(t)), && (1)\\
&\cdots && (2)\\
\dot{x}_i(t) &= f_i(t, x_1(t), \cdots, x_n(t)), && (3)\\
&\cdots && (4)\\
\dot{x}_n(t) &= f_n(t, x_1(t), \cdots, x_n(t)). && (5)
\end{aligned}
$$

and the initial conditions:

$$
\begin{aligned}
x_1(0) &= y_1, && (6)\\
&\cdots && (7)\\
x_i(0) &= y_i, && (8)\\
&\cdots && (9)\\
x_n(0) &= y_n. && (10)
\end{aligned}
$$

Often it is convenient to introduce more compact notations. First we combine all the space variables $x_1, \cdots, x_n \in \mathbb{R}$ into a single vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We also identify a vector $(t, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the vector $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Define a function $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by:

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^n \quad F(t, x) = (f_1(t, x), f_2(t, x), \cdots, f_n(t, x)). \tag{11}$$

The above system of (coupled) scalar ordinary differential equations can then be reformulated as finding one vector-valued differentiable function $x : I \to \mathbb{R}^n$ such that:

$$\dot{x}(t) = F(t, x(t)), \tag{12}$$

whereas the initial condition can be written in the form:

$$x(0) = y. \tag{13}$$

2 Making an ODE autonomous

The above non-autonomous system can be transformed into an autonomous one by the following trick. We introduce an additional function $x_0 : I \to \mathbb{R}$ and remark that the condition:

$$\forall t \in I \quad x_0(t) = t, \tag{14}$$

is equivalent to the conditions:

$$\forall t \in I \quad \dot{x}_0(t) = 1 \quad \text{and} \quad x_0(0) = 0. \tag{15}$$

Therefore the above system of scalar ordinary differential equations is equivalent to the problem of finding $n+1$ scalar functions $x_i : I \to \mathbb{R}$ (for $i \in \{0, 1, \cdots, n\}$) satisfying:

$$\begin{aligned}
\dot{x}_0(t) &= 1 && (16)\\
\dot{x}_1(t) &= f_1(x_0(t), x_1(t), \cdots, x_n(t)), && (17)\\
&\cdots && (18)\\
\dot{x}_i(t) &= f_i(x_0(t), x_1(t), \cdots, x_n(t)), && (19)\\
&\cdots && (20)\\
\dot{x}_n(t) &= f_n(x_0(t), x_1(t), \cdots, x_n(t)). && (21)
\end{aligned}$$

and the initial conditions:

$$\begin{aligned}
x_0(0) &= 0, && (22)\\
x_1(0) &= y_1, && (23)\\
&\cdots && (24)\\
x_i(0) &= y_i, && (25)\\
&\cdots && (26)\\
x_n(0) &= y_n. && (27)
\end{aligned}$$

In more compact notations this problem corresponds to the one of finding one vector-valued function $x : I \to \mathbb{R}^{n+1}$ (notice the dimension of the target space) satisfying the ODE:

$$\dot{x}(t) = G(x(t)), \tag{28}$$

where $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is defined by:

$$\forall x \in \mathbb{R}^{n+1} \quad G(x) = (1, F(x)), \tag{29}$$

as well as the initial condition:

$$x(0) = (0, y_1, \cdots, y_n). \tag{30}$$

Example 2.1 *The scalar IVP:*

$$\dot{y}(t) = t \sin y(t) \quad \textit{and} \quad y(0) = 1, \tag{31}$$

is equivalent to the 2-dimensional ODE:

$$\dot{x}(t) = 1, \tag{32}$$
$$\dot{y}(t) = x(t)\sin y(t), \tag{33}$$

with initial conditions:

$$x(0) = 0, \tag{34}$$
$$y(0) = 1. \tag{35}$$

3 Reducing the order of an equation

For simplicity we just consider scalar equations. Fix an integer $k \geq 2$. Let $f : \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function, and pick $y \in \mathbb{R}^k$. We consider the initial value problem of finding an interval $I \subset \mathbb{R}$ containing 0 and a k times differentiable function $x : I \to \mathbb{R}$ such that:

$$\forall t \in I \quad x^{(k)}(t) = f(t, x^{(0)}(t), x^{(1)}(t), \cdots, x^{(k-1)}(t)), \tag{36}$$

where $x^{(i)}$ denotes the i-th derivative of x, as well as the initial condition :

$$(x^{(0)}(0), x^{(1)}(0), \cdots, x^{(k-1)}(0)) = y. \tag{37}$$

This IVP can be transformed into an IVP for a vector-valued function involving only first order derivatives, as follows. First introduce k functions $z_i : I \to \mathbb{R}$. We notice that the condition:

$$\forall i \in \{0, \cdots, k-1\} \quad z_i = x^{(i)}, \tag{38}$$

is equivalent to the conditions:

$$z_0 = x \quad \text{and} \quad \forall i \in \{0, \cdots, k-2\} \quad \dot{z}_i = z_{i+1}. \tag{39}$$

Therefore the initial value problem for the function x is equivalent to saying that x should be the first component of a vector-valued function $z : I \to \mathbb{R}^k$ satisfying:

$$\dot{z}_0(t) = z_1(t), \tag{40}$$
$$\cdots \tag{41}$$
$$\dot{z}_{k-2}(t) = z_{k-1}(t), \tag{42}$$
$$\dot{z}_{k-1}(t) = f(t, z_0(t), \cdots, z_{k-1}(t)), \tag{43}$$

and the initial conditions:

$$(z_0(0), \cdots, z_{k-1}(0)) = y. \tag{44}$$

To obtain more compact notations we can introduce the function $F : \mathbb{R}^{k+1} \to \mathbb{R}^k$ defined by for all $(t, z_0, \cdots, z_{k-1}) \in \mathbb{R}^{k+1}$:

$$F(t, z_0, \cdots, z_{k-1}) = (z_1, \cdots, z_{k-1}, f(t, z_0, \cdots, z_{k-1})). \tag{45}$$

Then the IVP is:

$$\dot{z}(t) = F(t, z(t)) \quad \text{and} \quad z(0) = 0. \tag{46}$$

Notice that sometimes we use z_i to denote a function $I \to \mathbb{R}$ and sometimes to denote an element of $\mathbb{R}$. This is of course *very bad* but most people do it.

Example 3.1 *The second order scalar equation:*

$$\ddot{\theta}(t) = -\sin\theta(t) \quad \textit{and} \quad \theta(0) = 0,\ \dot{\theta}(0) = 1, \tag{47}$$

is equivalent to finding the first component of the 2-dimensional system:

$$\dot{x}(t) = y(t), \tag{48}$$
$$\dot{y}(t) = -\sin x(t), \tag{49}$$

with initial conditions:

$$x(0) = 0, \tag{50}$$
$$y(0) = 1. \tag{51}$$