Notes for MAT-INF1310 – 3
Snorre Christiansen, February 3, 2005

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1 Gronwall's lemma, Lipschitz continuity and uniqueness of solutions

The following lemma is due to Gronwall.

Lemma 1.1 *Let I be an interval in $\mathbb{R}$. Let a and b be two continuous functions $I \to \mathbb{R}$. Pick $t_0 \in I$ and x_0 in $\mathbb{R}$. Suppose $x : I \to \mathbb{R}$ is a solution of :*

$$\dot{x}(t) = a(t)x(t) + b(t) \quad \textit{for} \quad t \geq t_0, \tag{1}$$
$$x(t_0) = x_0. \tag{2}$$
$$\tag{3}$$

Suppose that $y : I \to \mathbb{R}$ is a continuously differentiable function such that:

$$\dot{y}(t) \leq a(t)y(t) + b(t), \quad \textit{for} \quad t \geq t_0, \tag{4}$$
$$y(t_0) \leq x_0. \tag{5}$$
$$\tag{6}$$

Then for all $t \in I$ such that $t \geq t_0$, $y(t) \leq x(t)$.

Briefly put, a proof consists in substracting (1) from (4), putting all terms on the left hand side, multiplying by an integrating factor and integrating from t_0 to t. The most frequent use of this lemma is the following special case:

Example 1.1 *Suppose $y : [t_0, t_1) \to \mathbb{R}$ is a continuously differentiable function such that:*

$$\dot{y}(t) \leq My(t) \quad \textit{for} \quad t \geq t_0. \tag{7}$$

Then for all $t \geq t_0$, $y(t) \leq y(t_0)e^{M(t-t_0)}$.

This example is the object of Rob. exercise 9.7 (independently of the general lemma stated above).

We equip $\mathbb{R}^n$ with the Euclidean norm $\| \cdot \|$ defined by :

$$\forall (x_1, \cdots, x_n) \in \mathbb{R}^n \quad \|(x_1, \cdots, x_n)\| = (x_1^2 + \cdots + x_n^2)^{1/2}. \tag{8}$$

Proposition 1.2 *Suppose I is an interval and U is an open subset of $\mathbb{R}^n$. Suppose $F : I \times U \to \mathbb{R}^n$ is a continuous function such that for some $L \geq 0$ we have:*

$$\forall t \in I \ \forall x, y \in U \quad \|F(t, x) - F(t, y)\| \leq L\|x - y\|. \tag{9}$$

Then for any $t_0 \in I$ and $x_0 \in U$ and any interval J such that $t_0 \in J \subset I$, there is at most one solution on J to the initial value problem:

$$\dot{x} = F(t, x) \quad \textit{and} \quad x(t_0) = x_0. \tag{10}$$

Briefly put, a proof consists in supposing that we have two solutions x and y on J and defining a function $z : I \to \mathbb{R}$ by putting $z(t) = \|x(t) - y(t)\|^2$. Using in particular the Cauchy-Schwartz inequality, one proves an estimate like (7) for z and deduce that $z(t) = 0$ for $t \geq t_0$. To obtain a similar estimate for $t \leq t_0$ one writes down a differential equation for $t \mapsto x(-t)$ and $t \mapsto y(-t)$ and repeat the reasoning for these solutions.

Actually at very little extra cost one can prove:

Exercise 1.1 *Suppose that F is a function having the properties stated in the previous proposition. Suppose $t_0 \in I$ and that $x_0, y_0 \in U$. Suppose x and y are two solutions, on an interval J containing t_0, to the ordinary differential equation $\dot{z}(t) = F(t, z(t))$, with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$. Prove that :*

$$\forall t \in J \ t \geq t_0 \Rightarrow \|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{L(t-t_0)}. \tag{11}$$

A function $(t, x) \mapsto F(t, x)$ having the property (9) for some $L \geq 0$ is said to be uniformly Lipschitz with respect to the space variable x (and with Lipschitz constant L). Sometimes one just says that F is Lipschitz though this is an abuse[1].

We also proved that if $U \subset \mathbb{R}$ is an interval, and $(t, x) \mapsto F(t, x)$ is a function which is continuously differentiable with respect to x and $|\partial_x F(t, x)| \leq L$ for all $(t, x) \in I \times U$ then (9) holds. This result can be extended to higher dimensions (when $U \subset \mathbb{R}^n$ is a sufficiently nice subset, for instance a convex one[2]), and we will do that later in the semester.

Exercise 1.2 *Show that when $0 < \alpha < 1$, the function $x \mapsto x^\alpha$ is not Lipschitz on any interval $[0, a]$ for $a > 0$, but is Lipschitz on all intervals of the form $[a, +\infty)$ for $a > 0$.*

[1] A function $G : U \to V$ where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ is said to be Lipschitz when there is a constant $L \geq 0$ such that:

$$\forall x, y \in U \quad \|G(x) - G(y)\| \leq L\|x - y\|. \tag{12}$$

So the property we ask of $F : I \times U \to \mathbb{R}^n$ is weaker than being Lipschitz, but stronger than "$F(t, \cdot)$ is Lipschitz for each $t \in I$".

[2] A convex subset of $\mathbb{R}^n$ is a a subset U such that for any two points x_0 and y_0 in U the segment joining them is also included in U (i.e. for all $s \in [0, 1]$, $sx_0 + (1 - s)y_0 \in U$).