Notes for MAT-INF1310 – 5
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1 Midterm exam - suggestions for solutions

Exercise 1

a. (2 points) For all $t \in]0, \pi[$ we have:

$$\begin{aligned} f'(t) &= \log'(\tan(\frac{t}{2}))\tan'(\frac{t}{2})\frac{1}{2}, && (1)\\ &= \frac{1}{\tan(\frac{t}{2})}\frac{1}{(\cos(\frac{t}{2}))^2}\frac{1}{2}, && (2)\\ &= \frac{1}{2\sin(\frac{t}{2})\cos(\frac{t}{2})}, && (3)\\ &= \frac{1}{\sin(t)}. && (4) \end{aligned}$$

b. (6 points) An integrating factor I for this linear equation is given by:

$$I(t) = \exp(\log(\tan(\frac{t}{2})) = \tan(\frac{t}{2}). \quad (5)$$

Multiplying the differential equation by the integrating factor and integrating from $\pi/2$ to $t \in]0, \pi[$ gives :

$$x(t)\tan(\frac{t}{2}) - x_0 \tan(\frac{\pi}{4}) = \int_{\frac{\pi}{2}}^{t} \tan(\frac{s}{2})\mathrm{d}s. \quad (6)$$

Since $\tan(\pi/4) = 1$ and an antiderivative of tan on $]0, \pi/2[$ is $-\log\cos$ we get:

$$x(t)\tan(\frac{t}{2}) - x_0 = -2\log(\cos(\frac{t}{2})) + 2\log(\cos(\frac{\pi}{4})). \quad (7)$$

Since $\cos(\pi/4) = \sqrt{2}/2$ we obtain:

$$x(t) = \frac{x_0 - 2\log(\cos(\frac{t}{2})) - \log(2)}{\tan(\frac{t}{2})}. \quad (8)$$

c. (2 points) We have:

$$x(\frac{2\pi}{3}) = \frac{x_0 - 2\log(\cos(\frac{\pi}{3})) - \log(2)}{\tan(\frac{\pi}{3})}. \quad (9)$$

Since $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$ we obtain the desired result:

$$x(\frac{2\pi}{3}) = \frac{x_0 + \log(2)}{\sqrt{3}}. \quad (10)$$

Exercise 2 (5 points) For all $t \geq 0$ we have $x(t) \geq 0$ and $\sin(t^2) \leq 1$ hence:

$$\dot{x}(t) = \sin(t^2)x(t) \leq x(t). \tag{11}$$

Gronwall's lemma gives, for all $t \geq 0$:

$$x(t) \leq x(0)e^t. \tag{12}$$

Hence we have :

$$0 \leq e^{-2t}x(t) \leq e^{-t}x(0). \tag{13}$$

Since $e^{-t} \to 0$ when $t \to +\infty$ we obtain the desired result:

$$\lim_{t \to +\infty} e^{-2t}x(t) = 0. \tag{14}$$

Exercise 3

a. (3 points) For any $y \neq 0$ we have :

$$\begin{aligned} \frac{\phi(y) - \phi(0)}{y} &= \frac{\frac{\sin y}{y} - 1}{y} = \frac{\sin y - y}{y^2}, & (15) \\ &= \frac{\mathcal{O}(y^3)}{y^2} = \mathcal{O}(y). & (16) \end{aligned}$$

Hence ϕ is differentiable at 0 and:

$$\phi'(y) = 0. \tag{17}$$

At any point $y \neq 0$, ϕ is differentiable and:

$$\phi'(y) = \frac{y\cos(y) - \sin(y)}{y^2}. \tag{18}$$

The function ϕ' is continuous at all $y \neq 0$. Moreover for $y \neq 0$ we have:

$$\phi'(y) - \phi'(0) = \frac{y\cos(y) - \sin(y)}{y^2} = \frac{y(1 + \mathcal{O}(y^2)) - y + \mathcal{O}(y^3)}{y^2} = \mathcal{O}(y). \tag{19}$$

Hence ϕ' is also continuous at 0.

For this question you could also use Hopital's rule.

b. (3 points) A first method is to do the following computations:

For $t \neq 0$ we have:

$$\begin{aligned} f(t,x) &= \frac{xt\cos(xt) - \sin(xt)}{t^2}, & (20) \\ g(t,x) &= \cos(xt), & (21) \end{aligned}$$

hence:

$$\begin{aligned} \partial_x f(t,x) &= \frac{t\cos(xt) - xt^2\sin(xt) - t\cos(xt)}{t^2} = -x\sin(xt), & (22) \\ \partial_t g(t,x) &= -x\sin(xt). & (23) \end{aligned}$$

This shows that the equation is exact.

A second method is to directly try to find F such that:

$$\partial_t F(t,x) = f(t,x) \quad \text{and} \quad \partial_x F(t,x) = g(t,x). \tag{24}$$

The first condition is equivalent to:

$$F(t,x) = x\phi(xt) + C(x) = \frac{\sin(xt)}{t} + C(x). \tag{25}$$

Taking this into consideration, the second condition is equivalent to:

$$\cos(xt) + C'(x) = \cos(xt), \tag{26}$$

In order to obtain a suitable F it is enough to let C be any constant function. The existence of a suitable F shows that the equation exact.

In the second method, or in general to construct F, you could also start with trying to ensure the second condition, which is perhaps easier.

c. (4 points) Going through the second method for question b we are led to define F by the formula:

$$F(t,x) = x\phi(xt). \tag{27}$$

Then F is continuously differentiable and x is a solution to (2) if and only if $t \to F(t, x(t))$ is constant and $x(0) = x_0$. Hence if x is a solution we have for all t:

$$x(t)\phi(x(t)t) = F(t, x(t)) = F(0, x(0)) = x_0. \tag{28}$$

We therefore have ($x(t) \neq 0$ and) for $t \neq 0$:

$$x(t)\frac{\sin(x(t)t)}{x(t)t} = x_0, \tag{29}$$

which gives:

$$\sin(x(t)t) = x_0 t. \tag{30}$$

Since we must have $|\sin(x(t)t)| \leq 1$ a solution can only be defined for t such that $|x_0 t| \leq 1$. Since $x_0 \neq 0$ this requires $|t| \leq |x_0|^{-1}$.