Notes for MAT-INF1310 – 7
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1 Vandermonde

Given n real or complex numbers $a_0, a_1, \cdots, a_{n-1}$, the associated Vandermonde matrix is the n by n matrix A (indexed by the set $[\![0, n-1]\!]^2$) defined by $A_{ij} = a_j^i$. The associated Vandermonde determinant is the determinant of this matrix. We will denote it by $V(a_0, \cdots, a_{n-1})$. Thus:

$$V(a_0, \cdots, a_{n-1}) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_{n-1} \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} \end{pmatrix}. \tag{1}$$

Proposition 1.1 *We have:*

$$V(a_0, \cdots a_{n-1}) = \prod_{(i,j) \in [\![0,n-1]\!]^2 \ : \ i<j} (a_j - a_i). \tag{2}$$

– *Proof:* For $n = 1$ it is the definition of a product over an empty index set. The statement is also trivial for $n = 2$: it simply says that:

$$\det \begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix} = a_1 - a_0. \tag{3}$$

For greater n proceed as follows: For rows indexed by $i = n - 1$ down to $i = 1$ substract a_0 times row $i - 1$ from row i. Row $i = 0$ is unchanged. These operation do not change the value of the determinant. For i between $n - 1$ and 1, row i then looks like:

$$(0 \ \ (a_1 - a_0)a_1^{i-1} \ \ \cdots \ \ (a_{n-1} - a_0)a_{n-1}^{i-1}). \tag{4}$$

Column $j = 0$ is:

$$\begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}. \tag{5}$$

For $j \geq 1$ column j looks like:

$$\begin{pmatrix} 1 \\ (a_j - a_0)1 \\ (a_j - a_0)a_j \\ \cdots \\ (a_j - a_0)a_j^{n-2} \end{pmatrix}. \tag{6}$$

One gets:

$$V(a_0,\cdots,a_{n-1}) = V(a_1,\cdots,a_{n-1}) \prod_{j\in[\![0,n-1]\!]\ :\ 0<j} (a_j - a_0). \tag{7}$$

An induction argument completes the proof. □

2 Wronski

Given n (real or complex valued) functions $f_0, f_1, \cdots, f_{n-1}$ defined on some non-trivial interval[1] I, their Wronskian is the function on I denoted $W(f_0,\cdots,f_{n-1})$ and defined by, for each $t \in I$:

$$W(f_0,\cdots,f_{n-1})(t) = \det \begin{pmatrix} f_0(t) & f_1(t) & \cdots & f_{n-1}(t) \\ f_0'(t) & f_0'(t) & \cdots & f_{n-1}'(t) \\ f_0''(t) & f_1''(t) & \cdots & f_{n-1}''(t) \\ \cdots & \cdots & \cdots & \cdots \\ f_0^{(n-1)}(t) & f_1^{(n-1)}(t) & \cdots & f_{n-1}^{(n-1)}(t) \end{pmatrix}. \tag{8}$$

In other words $W(f_0,\cdots,f_{n-1})(t)$ is the determinant of the n by n matrix $F(t)$ defined by $F(t)_{ij} = f_j^{(i)}(t)$.

Example 2.1 *If $a_0,\cdots,a_{n-1}$ are n real (or complex) numbers and the function f_i is defined by $f_i(t) = \exp(a_i t)$, then we have:*

$$W(f_0,\cdots,f_{n-1})(t) = V(a_0,\cdots,a_{n-1}) \exp((a_0+\cdots+a_{n-1})t) \tag{9}$$

Proposition 2.1 *If the functions $f_0,\cdots,f_{n-1}$ are linearly dependent on a non-trivial interval I their Wronskian is 0 at each point of I.*

– *Proof:* Supposing they are linearly dependent, differentiate this linear relation up to order $n-1$ and deduce that the previously defined matrix $F(t)$ has linearly dependent columns. □

This proposition is often used the other way around. That is, to ensure that functions $f_0,\cdots,f_{n-1}$ are linearly independent, it is enough to find one t such that $W(f_0,\cdots,f_{n-1})(t) \neq 0$. For instance we obtain from Example 2.1:

Corollary 2.2 *If the numbers $a_0,\cdots,a_{n-1}$ are two by two distinct their Vandermonde determinant is non-zero, and the functions f_i defined by $f_i(t) = \exp(a_i t)$ for $t \in I$ are linearly independent (whenever the interval I is not trivial).*

3 Exercises

Exercise 3.1 *Suppose $f_0,\cdots,f_{n-1}$ are n real-valued functions on a non-trivial interval I with the property that for each $i \in [\![0,n-1]\!]$ there exists a $t_i \in I$ such that $f_i(t_i) \neq 0$ but for $j \neq i$ we have $f_j(t_i) = 0$. Show that the functions f_i are linearly independent. Try to find some examples of functions with these properties.*

[1] In these notes a non-trivial interval is an interval containing at least two points.

Exercise 3.2 *Find two linearly independent real-valued functions f and g defined on $\mathbb{R}$ which are differentiable, and such that $W(f,g)$ is 0 at each point. You can use the preceding exercise.*

Exercise 3.3 *Pick n functions $f_1, \cdots, f_n$ on an interval I with derivatives of all orders. Suppose their Wronskian is non-zero at each point. Consider the equation with unknown function f : $W(f_1, \cdots, f_n, f) = 0$. This is a linear differential equation. What is the order? What is the coefficient correponding to the highest order of derivation? Find a basis for the space of solutions.*

Exercise 3.4 *Suppose a and b are two continuous (real or complex-valued) functions defined on an interval I. Suppose f and g are two solutions of the second order differential equation:*

$$h''(t) + a(t)h'(t) + b(t)h(t) = 0. \tag{10}$$

Show (by direct computation) that $W(f,g)$ satisfies the first order differential equation:

$$h'(t) + a(t)h(t) = 0. \tag{11}$$

Can you see a relationship between this Exercise and Example 2.1 in the case $n = 2$?

4 Operators

Given a vector space V an operator on V is a linear map $A : V \to V$. If A is an operator and λ is a scalar (a real or complex number according to the type of vector space) λA is defined to be the operator:

$$\lambda A : x \mapsto \lambda(Ax). \tag{12}$$

If A and B are two operators, their sum $A + B$ is defined to be the operator:

$$A + B : x \mapsto (Ax) + (Bx), \tag{13}$$

and their product AB is defined to be the operator:

$$AB : x \mapsto A(B(x)). \tag{14}$$

If you are unfamiliar with these definitions you should check that they really define operators (*linear* maps $V \to V$).

Given a polynomial P defined by $P(X) = \sum_k p_k X^k$, and an operator $A : V \to V$ the operator $P(A)$ is defined to be the operator $\sum_k p_k A^k$.

Exercise 4.1 *Check that if λ is a scalar, P and Q are two polynomials and $A : V \to V$ is an operator we have: $(\lambda P)(A) = \lambda(P(A))$, $(P + Q)(A) = P(A) + Q(A)$ and $(PQ)(A) = P(A)Q(A)$. Of these three identities the last is the most interesting; notice that on the left we have a product of polynomials evaluated on an operator and on the right we have a product of operators.*

Example 4.1 *Let I be a non-trivial interval. The space of real valued functions on I is a vector space; denote it by $\mathbb{R}^I$. Suppose $a : I \to \mathbb{R}$ is a real-valued function on I. The map which to any function u on I associates the function v defined by $v(t) = a(t)u(t)$ is an operator on $\mathbb{R}^I$ (check it!). In other words if we call this operator A, we have $(Au)(t) = a(t)u(t)$. This operator is called* multiplication by a.

Example 4.2 *Fix a real τ. Suppose u is a real-valued function defined on $\mathbb{R}$. Define $T_\tau u$ to be the function defined for $t \in \mathbb{R}$ by $(T_\tau u)(t) = u(t - \tau)$. Then T_τ is an operator on the vectorspace of real-valued functions defined on $\mathbb{R}$. This operator is called the* translation operator.

Exercise 4.2 *Check that if σ and τ are reals, $T_\sigma T_\tau = T_{\sigma+\tau}$ and that $T_\tau^n = T_{n\tau}$ for any $n \in \mathbb{N}$.*

Exercise 4.3 *Fix a complex a and a real τ. Let $u : \mathbb{R} \to \mathbb{C}$ be the function defined by $u(t) = \exp(at)$. Show that for any polynomial P we have : $P(T_\tau)u = P(\exp(-a\tau))u$ (notice that on the left we apply an operator to u, whereas on the right we multiply u by a complex number). You can start by looking at the case where $P(X)$ is of the form X^n.*

Example 4.3 *Let I be an interval. Let V be the vector space of all functions on I which are differentiable arbitrarily many times. The* derivation operator *on V is the operator $D : u \to u'$.*

Exercise 4.4 *Fix a complex a. Let $u : \mathbb{R} \to \mathbb{C}$ be the function defined by $u(t) = \exp(at)$. Show that for any polynomial P we have: $P(D)u = P(a)u$.*

Exercise 4.5 *For any function $a : I \to \mathbb{R}$, let M_a be the operator defined as multiplication by a, as in Example 4.1. Suppose that a can be differentiated arbitrarily many times. Prove that if D is the derivation operator as in Example 4.3 we have:*

$$M_a D - D M_a = -M_{a'}. \tag{15}$$