Notes for MAT-INF1310 – 8
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1 High order equations and systems

Exercise 1.1 *Pick $k \geq 1$ and reals $a_{k-1}, \cdots, a_0$. Recall that the k-th order linear equation:*

$$x^{(k)}(t) + a_{k-1}x^{(k-1)}(t) + \cdots + a_0 x(t) = 0, \tag{1}$$

can be analysed via the system of k first order equations:

$$X'(t) = AX(t), \quad A = \begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & 0 & 0 \\ 0 & & \ddots & \ddots & 0 \\ & & & 0 & 1 \\ c_0 & c_1 & \cdots & c_{k-2} & c_{k-1} \end{pmatrix}, \tag{2}$$

with $c_l = -a_l$ for each l.

Prove (by induction on k and development of a determinant along the first column) that for each λ:

$$\det(\lambda I - A) = \lambda^k + \sum_{l=0}^{k-1} a_l \lambda^l. \tag{3}$$

With this exercise the methods for solving (1) with the help of the characteristic polynomial, can be interpreted as a special case of the eigenvalue method for systems such as (2).

2 Complex and matrix exponentials

Exercise 2.1 *Let $f : \mathbb{C} \to M_2(\mathbb{R})$ be the map such that for each $a, b \in \mathbb{R}$ we have:*

$$f(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{4}$$

Prove that for each $u, v \in \mathbb{C}$ we have:

$$f(u + v) = f(u) + f(v), \tag{5}$$

and (more technical):

$$f(uv) = f(u)f(v). \tag{6}$$

Use this to prove (at least formally) that for each $u \in \mathbb{C}$:

$$f(\exp(u)) = \exp(f(u)). \tag{7}$$

Deduce that for each $a, b \in \mathbb{R}$:

$$\exp \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}. \tag{8}$$

Use this to give a formula for the solution of differential equations of the form (with $a, b \in \mathbb{R}$ given):

$$x'(t) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} x(t). \tag{9}$$

3 Summer holiday pleasure

Exercise 3.1 *Pick $n \geq 1$ (you can start with $n = 2$ and $n = 3$) and a matrix $A \in M_n(\mathbb{R})$. Define $f : \mathbb{R} \to \mathbb{R}$ by:*

$$f(h) = \det(I + hA). \tag{10}$$

Prove that:

$$f'(0) = \mathrm{tr}(A). \tag{11}$$

Suppose $X : \mathbb{R} \to M_n(\mathbb{R})$ is a differentiable matrix valued function, such that $X(t)$ is invertible for each $t \in \mathbb{R}$. Show that for each t we have a Taylor expansion for small h of the form:

$$X(t + h) = X(t)(I + hX(t)^{-1}X'(t) + \mathcal{O}(h^2)). \tag{12}$$

Define a function $g : \mathbb{R} \to \mathbb{R}$ by:

$$g(t) = \det(X(t)). \tag{13}$$

Show that we have:

$$g'(t) = \det(X(t))\,\mathrm{tr}(X(t)^{-1}X'(t)). \tag{14}$$

Suppose that in fact X is a solution to the linear differential equation:

$$X'(t) = A(t)X(t). \tag{15}$$

for some continuous matrix valued function $A : \mathbb{R} \to M_n(\mathbb{R})$. Show that g satisfies the linear differential equation:

$$g'(t) = \mathrm{tr}(A(t))g(t). \tag{16}$$

If you know about the comatrix of a matrix you can try to generalize this exercise to case of non invertible matrices $X(t)$...

Exercise 3.2 *Suppose that we are given n continuous functions $a_l : \mathbb{R} \to \mathbb{R}$ for $l = 0, \cdots, n-1$. Consider the differential equation:*

$$x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) = 0, \tag{17}$$

Transform this into a first order system of equations and use the preceding exercise to show that if $x_1, \cdots, x_n$ are n solutions to (17), then their Wronskian W satisfies the differential equation:

$$W'(t) + a_{n-1}(t)W(t) = 0. \tag{18}$$