Notes for MAT-INF1310 – 9
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1 Final exam - suggestions for solutions

Exercise 1 Since $\exp(2t - x) = \exp(2t)\exp(-x)$ the equation is separable. We have:

$$x'(t)\exp(x(t)) = \exp(2t). \tag{1}$$

Integrating from 0 to t gives:

$$\exp(x(t)) - \exp(0) = 1/2(\exp(2t) - \exp(0)). \tag{2}$$

Hence:

$$\exp(x(t)) = 1/2(\exp(2t) + 1), \tag{3}$$

which gives:

$$x(t) = \log(1/2(\exp(2t) + 1)). \tag{4}$$

Exercise 2

a. For all $t \in \mathbb{R}$ we have:

$$x_1'''(t) + x_1''(t) + x_1'(t) + x_1(t) = -e^{-t} + e^{-t} - e^{-t} + e^{-t} = 0, \tag{5}$$

hence x_1 is a solution.

b. (H) is a third order linear differential equation whose highest order coefficient vanishes nowhere, hence V has dimension 3. The characteristic polynomial is P defined by:

$$P(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1. \tag{6}$$

From question (a) we infer that -1 is a root of P. Dividing $P(\lambda)$ by $\lambda + 1$ we obtain:

$$P(\lambda) = (\lambda + 1)(\lambda^2 + 1) = (\lambda + 1)(\lambda + i)(\lambda - i). \tag{7}$$

From this we deduce that a basis for V is given by the functions:

$$x_1, \text{ cos, and sin}. \tag{8}$$

Alternatively one can obtain the roots of P by using that, for $\lambda \neq 1$:

$$P(\lambda) = (\lambda^4 - 1)/(\lambda - 1), \tag{9}$$

and that $\lambda^4 - 1 = 0$ iff $\lambda \in \{i, -1, -i, 1\}$.

c. i is a simple root of the characteristic polynomial. We look for a particular solution x_P in the form:

$$x_P(t) = At\cos(t) + Bt\sin(t). \tag{10}$$

Then we have:

$$\begin{aligned} x'_P(t) &= A(\cos(t) - t\sin(t)) + B(\sin(t) + t\cos(t)), && (11)\\ x''_P(t) &= A(-2\sin(t) - t\cos(t)) + B(2\cos(t) - t\sin(t)), && (12)\\ x'''_P(t) &= A(-3\cos(t) + t\sin(t)) + B(-3\sin(t) - t\cos(t)). && (13) \end{aligned}$$

Summing we obtain:

$$\begin{aligned} & x'''_P(t) + x''_P(t) + x'_P(t) + x_P(t) && (14)\\ = & (-2A + 2B)\cos(t) + (-2A - 2B)\sin(t). && (15) \end{aligned}$$

The choice $A = -1/4$ and $B = 1/4$ gives the particular solution:

$$x_P(t) = -t/4\cos(t) + t/4\sin(t). \tag{16}$$

The general solution to equation (E) is:

$$x_P(t) = ae^{-t} + b\cos(t) + c\sin(t) - t/4\cos(t) + t/4\sin(t), \quad a, b, c \in \mathbb{R}. \tag{17}$$

Exercise 3

a. Use whatever method to obtain:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + \beta \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \quad \alpha, \beta \in \mathbb{R}. \tag{18}$$

b. We have:

$$\begin{aligned} E'(t) &= 2x(t)x'(t) + 2y(t)y'(t), && (19)\\ &= 2x(t)(-y(t)) + 2y(t)(x(t) + ay(t)), && (20)\\ &= 2ay(t)^2, && (21)\\ &\leq 0. && (22) \end{aligned}$$

Therefore E is decreasing.

c. The characteristic polynomial P_a of $A(a)$ is given by:

$$P_a(\lambda) = -\lambda(a - \lambda) + 1 = \lambda^2 - a\lambda + 1. \tag{23}$$

The discriminant Δ_a is:

$$\Delta_a = a^2 - 4. \tag{24}$$

If $|a| > 2$ we have two real roots:

$$r_\pm = (1/2)(a \pm \sqrt{a^2 - 4}). \tag{25}$$

Since $a^2 - 4 < a^2$ both roots are < 0 when $a < -2$, and both are > 0 when $a > 2$ (alternatively, the expression of P_a shows that the roots must have the same sign and that their sum is a, which gives – luckily – the same answer).

If $a = -2$ we have the double real root:

$$r = -1 < 0, \tag{26}$$

and if $a = 2$ we have the double real root:

$$r = 1 > 0. \tag{27}$$

If $|a| < 2$ we have two (distinct) complex roots:

$$r_\pm = (1/2)(a \pm i\sqrt{4 - a^2}). \tag{28}$$

They have strictly negative real part if and only if $a < 0$.

All in all the real part of the roots of P_a are all strictly negative if and only if $a < 0$.

To show that:

$$\lim_{t \to +\infty} x(t) = 0, \quad \lim_{t \to +\infty} y(t) = 0, \tag{29}$$

the idea is that the real part ρ of any eigenvalue gives rise to an exponential term of the form $t \mapsto \exp(\rho t)$ in the general solution, which is multiplied by other terms that are constant (distinct real roots), bounded (complex roots) or polynomial (double roots) in t. When $\rho < 0$ the exponential converges sufficiently rapidly to 0. More precisely:

If $a < -2$ the general solution of the linear system has the form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha e^{r_+ t} v_+ + \beta e^{r_- t} v_-. \tag{30}$$

with $r_+ < 0$ and $r_- < 0$.

If $a = -2$ one sees that the eigenspace associated with the eigenvalue -1 has dimension 1. The general solution of the linear system has the form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha e^{-t} v_1 + \beta e^{-t}(t v_1 + v_2). \tag{31}$$

If $0 > a > -2$ the general solution of the linear system has the form (with $\omega = (1/2)\sqrt{4 - a^2}$):

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{(a/2)t}(\alpha \cos(\omega t) v_+ + \beta \sin(\omega t) v_-). \tag{32}$$

In all three cases the convergence holds.

d. Putting $E(t) = x(t)^2 + y(t)^2$ we obtain as in (b):

$$\begin{aligned} E'(t) &= 2\sin(t^2) y(t)^2, && (33) \\ &\leq 2y(t)^2, && (34) \\ &\leq 2E(t). && (35) \end{aligned}$$

Hence Gronwall's lemma gives, for all $t \geq 0$:

$$E(t) \leq E(0)e^{2t} = e^{2t}. \tag{36}$$

This shows that:

$$x(t)^2 \leq e^{2t}, \quad \text{hence } |x(t)| \leq e^t. \tag{37}$$

We have shown that $C = 1$ is an adequate choice (it's the smallest possible one – why?).