1. obligatory exercise in MAT-INF 1310, Spring 2006

Deadline of delivery: Friday 17. March, 14:30.

Problem 1. Solve the initial value problem

$$\frac{dx}{dt} = x' = \frac{t+1}{2x}, \qquad x(0) = 1$$

Problem 2. If a particle with mass m moves in one dimension and has the position $x(t)$ at the time t, the Newton law of motion

$$mx'' = F(t, x, x')$$

is valid, where m is the mass, $x'' = \frac{d^2x}{dt^2}$ is the acceleration and the force F is a function of the time t, the position x and the velocity x'.

(i) First assume that F is a force of fricton proportional to a positive power α of the velocity x' and that F is growing linearly with time, i.e.

$$F(t, x, x') = -t(x')^{\alpha}$$

Assume in addition that $x(0) = 0$ and $x'(0) = 1$ and that $m = 1$.
Show that the velocity $x'(t)$ converges to 0 for all $\alpha \geq 1$. Show that there is a constant α_0 such that $\lim_{t\to\infty} x(t)$ is finite when $1 \leq \alpha < \alpha_0$ but $\lim_{t\to\infty} x(t) = +\infty$ when $\alpha_0 \leq \alpha$. Show that if $0 < \alpha < 1$, then the velocity will decrease to 0 after a finite time $t_0 > 0$ and compute t_0. Show that the solution of the differential equation is not unique in this case but it is a unique solution which is constant for $t \geq t_0$.

(In this problem you will find expressions for $x(t)$ involving integrals which cannot be computed explicitely but nevertheless the questions may be answered.)

(ii) If the force F er independent of t, Newton´s law of motion takes the form

$$mx'' = F(x, x')$$

and if in this case the velocity $y = x'$ is a function of x, $y = y(x)$, show that

$$(*) \qquad my\frac{dy}{dx} = F(x, y)$$

Use this to find $x(t)$ if

$$F(x, x') = -x(x')^2 + x^2(x')^4$$

and

$$x(0) = 0 \qquad \text{og} \qquad x'(0) = 1$$

(iii) If F depends only on the position x we say that F er *conservative* and we introduce the potential $V(x) = -\int\limits_0^x F(x)dx$.
In that case, if $y = y(x)$ is a solution of $(*)$ then

$$\tfrac{1}{2}my(x)^2 + V(x)$$

is independent of x. (The constant value E of this expression is called the total energy of the particle. $\frac{1}{2}my^2$ is the kinetic energy and $V(x)$ the potential energy.)

(iv) Adopt the conditions in (iii). Show that

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - V(x)}$$

Describe how this can be used to find $x(t)$ if $x(0)$ and $x'(0)$ are given, and use this method to find $x(t)$ if $m = 2$, $E = 1$ and $V(x) = x^2 =$ the harmonic oscillator potential. assume that $x(0) = 0$ and $x'(0) > 0$.

Problem 3. In this problem we will consider the coupled system

$$x' = x^2 - y^2 \qquad y' = x - y$$

of differential equations, where $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$

(i) Use Picards methiod on pp. 680–681 in EP to compute the three first approximations $(x_0(t), y_0(t)), (x_1(t)), y_1(t)),\ x_2(t), y_2(t))$ to the solutions for a general initial condition $x(0) = b_1,\ y(0) = b_2$.

(ii) Find all constant solutions of the system, i.e. find all $(a, b) \in \mathbb{R}^2$ such that the pair $x(t) = a,\ y(t) = b$ of constant functions is a solution.

(iii) Plot the direction field of this system (i.e, plot the vector (x', y') as a function of (x, y)) using a mathematics program like `Maple` or `Matlab` (see p.p. 479–480 i EP). State which commands you use in the program

(iv) Using

$$\frac{dx}{dy} = \frac{x'}{y'}$$

we obtain a first order differential equation for x as a function of y. Solve this equation and plot the graphs of some solution curves. Compare the results with the results from (iii).

(v) Plot and give a description of what happens to the solution curve starting in $\left(\frac{1}{4}, 0\right)$ at $t = 0$ when $t \to +\infty$ and $t \to -\infty$.

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