MAT-INF1310, Spring 2009
Mandatory Assignment 1
Deadline: March 6, 14:30.

This document contains guidelines for how to solve the problems posed in the first mandatory assignment. The solutions below might lack arguments needed in a complete solution.

1. Find a solution of the differential equation

$$\frac{dy}{dx} = 2xy + 3x^3 e^{x^2}$$

which satisfies $y(\ln 2) = 0$.

Solution: We want to solve the differential equation

$$y' - 2xy = 3x^3 e^{x^2}.$$

For this purpose we use an integrating factor T given by

$$T(x) = \exp\left(\int (-2x)dx\right) = e^{-x^2}.$$

Multiplying both sides of the equation by the integrating factor T gives

$$\frac{d}{dx}\left(e^{-x^2} y\right) = e^{-x^2} y' - 2xe^{-x^2} y = 3x^3.$$

Next, we integrate both sides in this equality with respect to x and get

$$e^{-x^2} y = \frac{3}{4}x^4 + C.$$

This gives us a general solution $y(x) = (3x^4/4 + C)e^{x^2}$ of the equation $y' - 2xy = 3x^3 e^{x^2}$. In order to have $y(\ln 2) = 0$, it is required that $C = -3(\ln 2)^4/4$. Thus, the solution of the initial value problem is

$$y(x) = \frac{3}{4}\big(x^4 - (\ln 2)^4\big)e^{x^2}.$$

□

2. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

Solution: We want to solve the differential equation

$$\frac{dP}{dt} = (\beta - \delta)P,$$

with $\beta = 0$ and $\delta = C/\sqrt{P}$, $C > 0$ constant. This gives the equation

$$P' = -C\sqrt{P}$$

which is separable. We get

$$\frac{P'}{\sqrt{P}} = -C$$

and integrating both sides with respect to t gives

$$\int \frac{P'(t)}{\sqrt{P(t)}}\,dt = \int \frac{1}{\sqrt{P}}\,dP = 2\sqrt{P} = -Ct + D.$$

Thus, $\sqrt{P(t)} = (D - Ct)/2$. Using the initial condition $P(0) = 900$ we get

$$30 = \sqrt{900} = \sqrt{P(0)} = (D - C \cdot 0)/2 = D/2,$$

and therefore $D = 60$. Using the initial condition $P(6) = 441$ we get

$$21 = \sqrt{441} = \sqrt{P(6)} = (60 - 6C)/2$$

and therefore $C = 3$. We have $\sqrt{P(t)} = (60 - 3t)/2$ and therefore

$$P(t) = \frac{(60 - 3t)^2}{4}.$$

It remains to find t such that $P(t) = 0$. We get

$$0 = P(t) = \frac{(60 - 3t)^2}{4} \quad \Longleftrightarrow \quad 0 = 60 - 3t,$$

i.e., $t = 20$. Thus, it took 20 weeks for all the fish to die. □

3. Find *all* solutions of the differential equation $y'' - 4y' + 9y = xe^x$.

Solution: We find the complementary functions by first observing that the characteristic polynomial of the equation is

$$r^2 - 4r + 9 = (r - 2)^2 - 2^2 + 9 = (r - 2)^2 + 5$$

and which has complex roots $r = 2 \pm i\sqrt{5}$. Thus, all complementary functions are given by

$$y_c(x) = e^{2x}\Big(c_1 \cos\big(\sqrt{5}\,x\big) + c_2 \sin\big(\sqrt{5}\,x\big)\Big), \qquad c_1, c_2 \in \mathbb{R}.$$

In order to find a particular solution, we first observe that the function $f(x) = xe^x$ and all of its derivatives can be described as linear combinations of the linearly independent functions e^x and xe^x. Indeed, $f^{(n)}(x) = ne^x + xe^x$. Since the functions e^x and xe^x are not among the complementary functions it is enough to determine the coefficients $a, b \in \mathbb{R}$ such that

$$y_p(x) = (a + bx)e^x$$

is a particular solution, see Rule 1 of the method of underdetermined coefficients. We get

$$y_p'(x) = be^x + (a + bx)e^x = be^x + y_p(x) \qquad \text{and} \qquad y_p''(x) = be^x + y_p'(x) = 2be^x + y_p(x).$$

Substituting y_p for y in the original equation gives

$$y_p'' - 4y_p' + 9y_p = (2be^x + y_p) - 4(be^x + y_p) + 9y_p = 6y_p - 2be^x = (6a - 2b)e^x + 6bxe^x$$

which is equal to xe^x if and only if $a = 1/18$ and $b = 1/6$. We conclude that any solution of the original equation is of the form

$$y(x) = \frac{1}{18}(1+3x)e^x + e^{2x}\Big(c_1\cos(\sqrt{5}\,x) + c_2\sin(\sqrt{5}\,x)\Big), \qquad c_1, c_2 \in \mathbb{R}.$$

□

4. Let y_1 and y_2 denote two linearly independent solutions of the differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous functions on an open (non-empty) interval I containing the point a. Suppose that Y is a third solution of the equation. This means that there are unique numbers $c_1, c_2 \in \mathbb{R}$ for which

$$Y(x) = c_1y_1(x) + c_2y_2(x), \qquad x \in I.$$

Express the values of the constants c_1 and c_2 in terms of the values of the functions y_1, y_2, and Y (and their derivatives) at the point a.

Solution: It is necessary that we find constants $c_1, c_2 \in \mathbb{R}$ such that

$$Y(a) = c_1y_1(a) + c_2y_2(a) \qquad \text{and} \qquad Y'(a) = c_1y_1'(a) + c_2y_2'(a),$$

i.e., we want to solve the linear equation

$$\begin{pmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} Y(a) \\ Y'(a) \end{pmatrix}.$$

Since y_1 and y_2 are linearly independent, we know that the Wronskian of y_1 and y_2 at the point a is non-zero. This means the (2×2)-matrix above is invertible. Indeed, we get

$$\begin{pmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{pmatrix}^{-1} = \frac{1}{y_1(a)y_2'(a) - y_1'(a)y_2(a)}\begin{pmatrix} y_2'(a) & -y_2(a) \\ -y_1'(a) & y_1(a) \end{pmatrix}$$

We get

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{y_1(a)y_2'(a) - y_1'(a)y_2(a)}\begin{pmatrix} y_2'(a) & -y_2(a) \\ -y_1'(a) & y_1(a) \end{pmatrix}\begin{pmatrix} Y(a) \\ Y'(a) \end{pmatrix}.$$

Carrying out the necessary products above, one gets

$$c_1 = \frac{y_2'(a)Y(a) - y_2(a)Y'(a)}{y_1(a)y_2'(a) - y_1'(a)y_2(a)} \qquad \text{and} \qquad c_2 = \frac{y_1(a)Y'(a) - y_1'(a)Y(a)}{y_1(a)y_2'(a) - y_1'(a)y_2(a)}. \tag{1}$$

Using these values of c_1 and c_2 we get that the two functions $Y(x)$ and $c_1y_1(x) + c_2y_2(x)$ are solutions of the same differential equation and satisfy the same initial value conditions. According to the theorem for existence and uniqueness of solutions of linear differential equations with initial values (Theorem 2 on page 104 in Edwards & Penney) there is a unique such function. Hence, the functions $Y(x)$ and $c_1y_1(x) + c_2y_y(x)$ have to agree on the interval I. This proves that the constants c_1 and c_2 calculated in (1) are such that

$$Y(x) = c_1y_1(x) + c_2y_2(x), \qquad x \in I.$$

□

5. Verify that if c is a constant, then the function defined by

$$y(x) = \begin{cases} 1 & \text{if } x \le c, \\ \cos(x-c) & \text{if } c < x < c+\pi, \\ -1 & \text{if } x \ge c+\pi \end{cases}$$

satisfies the differential equation $y' = -\sqrt{1-y^2}$ for all x. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has.

Solution: It is clear that the function y is differentiable on the open set $\mathbb{R} \setminus \{c, c+\pi\}$. It needs to be verified that y is differentiable at the points c and $c+\pi$, i.e., we need to verify that the limits defining $y'(c)$ and $y'(c+\pi)$ exist. These limits are given by

$$y'(c) = \lim_{h\to 0} \frac{y(c+h) - y(c)}{h} \qquad \text{and} \qquad y'(c+\pi) = \lim_{h \to 0} \frac{y(c+\pi+h) - y(c+\pi)}{h}.$$

Let us concentrate on the first limit. Observe that, since $0 < \cos(h) < 1$ for positive h close to zero, we get

$$\frac{\cos(h) - 1}{h} \le \frac{y(c+h) - y(c)}{h} \le 0$$

for h close to zero. Observe that

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = \lim_{h\to 0} \frac{\cos(0+h) - \cos(0)}{h} = \left(\frac{d}{dt}\cos t\right)(0) = -\sin(0) = 0.$$

This gives

$$0 = \lim_{h\to 0} \frac{\cos(h)-1}{h} \le \lim_{h\to 0} \frac{y(c+h)-y(c)}{h} \le 0,$$

which proves that $y'(c)$ exists and $y'(c) = 0$. Similarly it is proved that $y'(c+\pi) = 0$. From this it follows that

$$y'(x) = \begin{cases} 0 & \text{if } x \le c, \\ -\sin(x-c) & \text{if } c < x < c+\pi, \\ 0 & \text{if } x \ge c. \end{cases}$$

Let us now confirm that the function y satisfies the differential equation in question for any real number c. We get

$$-\sqrt{1-y^2(x)} = -\sqrt{1-(\pm 1)^2} = 0 = y(x), \qquad x \le c \quad \text{or} \quad x \ge c,$$

and for $c < x < c+\pi$ we get

$$-\sqrt{1-y^2(x)} = -\sqrt{1-\cos^2(x-c)} = -\sqrt{\sin^2(x-c)} = -\left|\sin^2(x-c)\right|$$

Observe that $0 \le \sin(x-c) \le 1$ for $c < x < c+\pi$. Hence, for such x we have $|\sin(x-c)| = \sin(x-c)$ and

$$-\sqrt{1-y^2(x)} = \begin{cases} 0 & \text{if } x \le c, \\ -\sin(x-c) & \text{if } c < x < c+\pi, \\ 0 & \text{if } x \ge c. \end{cases}$$

Thus it is proved that $y'(x) = -\sqrt{1-y^2(x)}$ for all $x \in \mathbb{R}$.

Next, we consider initial value conditions $y(a) = b$, for $a, b \in \mathbb{R}$. Since $-1 \le y(x) \le 1$ for all $x \in \mathbb{R}$, the initial value problem is unsolvable if $|b| > 1$.

Suppose that $b = 1$. Then, by the definition of the function y, $y(a) = b$ for all c such that $a \le c$. This means that we get infinitely many solutions. Similarly, if $b = -1$ we have that $y(a) = b$ for all c with $c \le a - \pi$. Thus, for $|b| = 1$ we find infinitely many solutions.

Finally, suppose that $|b| < 1$. There is a unique $\alpha \in \mathbb{R}$ with $0 < \alpha < \pi$ and $\cos(\alpha) = b$. By the definition of y we get that $y(a) = b$ if c is such that

$$a - c = \alpha + 2\pi n \qquad \text{and} \qquad c < a < c + \pi, \qquad n \in \mathbb{Z}.$$

These two conditions imply that $c < a = c + \alpha + 2\pi n < c + \pi$ is satisfied if and only if $n = 0$. Thus, for $c = a - \alpha$ the function y satisfies $y(a) = b$ and $c < a < c + \pi$. This shows that for $|b| < 1$ there is a unique solution satisfying $y(a) = b$. □

6. Suppose that the mass in a mass–spring–dashpot system with $m = 10$, $c = 9$ and $k = 2$ is set in motion with $x(0) = 0$ and $x'(0) = 5$ (see Figure 2.4.1 on page 135 in Edwards & Penney). Find how far the mass moves to the right before starting back toward the origin.

Solution: We want to find the unique solution satisfying

$$10x'' + 9x' + 2x = 0, \qquad x(0) = 0, \qquad x'(0) = 5.$$

Note that we might just as well consider the equation $x'' + 9x'/10 + 2x/10 = 0$. We start by finding the complementary functions y_c of this equation. The characteristic polynomial $r^2 + 9r/10 + 2/10$ can be written as

$$r^2 + \frac{9}{10}r + \frac{2}{10} = \left(r + \frac{9}{20}\right)^2 - \left(\frac{9}{20}\right)^2 + \frac{2}{10} = \left(r + \frac{9}{20}\right)^2 - \frac{1}{400}.$$

Thus its roots are given by

$$r = -\frac{9}{20} \pm \frac{1}{20}$$

and the roots are $r_1 = -2/5$ and $r_2 = -1/2$. This gives a general solution

$$x(t) = Ae^{-2t/5} + Be^{-t/2}.$$

Using the initial values we get

$$x(0) = A + B = 0 \qquad \text{and} \qquad x'(0) = -2A/5 - B/2 = 5$$

which implies $A = -B = 50$. The solution we seek is thus $x(t) = 50(e^{-2t/5} - e^{-t/2})$.

In order to find the maximum value of x we seek the smallest positive t for which the function x attains a maximum. For this reason we study the equation $x'(t) = 0$ and get

$$0 = x'(t) = 50\left(-\frac{2}{5}e^{-2t/5} + \frac{1}{2}e^{-t/2}\right) \qquad \Longrightarrow \qquad e^{-2t/5} = \frac{5}{4}e^{-t/2}.$$

Now, multiplying this equation by $e^{t/2}$ we get

$$e^{t/2}e^{-2t/5} = e^{t/10} = \frac{5}{4} \qquad \Longrightarrow \qquad \frac{t}{10} = \ln\left(\frac{5}{4}\right),$$

i.e., $t = 10\ln(5/4)$. One can confirm that this is a maximum of x by checking that $x''(t) < 0$ for this value of t. Moreover, observe that this is the only positive t for which the derivative is zero.

To find the position we calculate the value of x at time $t = 10\ln(5/4)$:

$$x\big(10\ln(5/4)\big) = 50\left(e^{-4\ln(5/4)} - e^{-5\ln(5/4)}\right) = 50\left(\left(e^{\ln(4/5)}\right)^4 - \left(e^{\ln(4/5)}\right)^5\right)$$
$$= 50\left(\left(\frac{4}{5}\right)^4 - \left(\frac{4}{5}\right)^5\right) = 2\cdot 5^2\left(\frac{4^4\cdot 5}{5^5} - \frac{4^5}{5^5}\right) = \frac{2^9}{5^3} = \frac{2^{12}}{2^3\cdot 5^3} = \frac{4096}{1000} = 4.096.$$

Thus, the box moves 4.096 to the right of the equilibrium before turning back.

7. Let p, q and r be continuous functions on some open (non-empty) interval I. Prove that the equation

$$y^{(3)} + p(x)y'' + q(x)y' + r(x)y = 0$$

has three solutions on the interval I which are linearly independent.

Solution: According to the theorem for existence and uniqueness of solutions to linear differential equations (Theorem 2 on page 114 in Edwards & Penney), there exists three solutions y_1, y_2, and y_3 satisfying

$$y_1(a) = y_2'(a) = y_3''(a) = 1$$

and

$$y_1'(a) = y_1''(a) = y_2(a) = y_2''(a) = y_3(a) = y_3'(a) = 0.$$

Evaluating the Wronskian $W = W(y_1, y_2, y_3)$ at the point a we get

$$W(a) = \det\begin{pmatrix} y_1(a) & y_2(a) & y_3(a) \\ y_1'(a) & y_2'(a) & y_3'(a) \\ y_1''(a) & y_2''(a) & y_3''(a) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \neq 0,$$

which proves (according to Theorem 3 on page 119 in Edwards & Penney) that the three solutions y_1, y_2, and y_3 are linearly independent. □