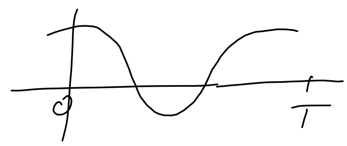
Kap. 2

Hva hvis vi i stedet for har en digital lyd

$f(t) =$ 

$\vec{x} =$ 

Hvordan splitter vi opp $\vec{x}$ som en sum av (digitale) rene toner?

Hva er "diskret Fourieranalyse"?

f def på $[0, T]$ $\longleftrightarrow$ $\vec{x} = (x_0, x_1, \dots, x_{N-1}) \in \mathbb{R}^N$

$\{\phi_n\}_{n=0}^{N-1}$ er en ortonormal basis:

$$\langle \phi_n, \phi_m \rangle = \sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi i k n/N} \frac{1}{\sqrt{N}} e^{2\pi i k m/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k(n-m)/N}$$ → Detter er en geom. rekke med $k = e^{2\pi i (n-m)/N}$

$$= \begin{cases} n = m: \frac{1}{N}\sum_{k=0}^{N-1} 1 = 1 \\ n \neq m: \frac{1}{N} a_0 \frac{1-k^N}{1-k} \end{cases}$$

$$1 - k^N = 1 - (e^{2\pi i(n-m)/N})^N = 1 - e^{2\pi i (n-m)} = 1 - 1 = 0$$

$\Rightarrow$ ortonormal basis

Fouriermatrisen er unitær (dvs. $F_N^H F_N = I$)

F_N: koord. skiftet $\{e_n\}_{n=0}^{N-1} \rightarrow \{\phi_n\}_{n=0}^{N-1}$

F_N^{-1} koord. skiftet $\{\phi_n\}_{n=0}^{N-1} \rightarrow \{e_n\}_{n=0}^{N-1}$

$\uparrow$

"søylene er gammel basis uttrykt i ny basis".

$$\phi_{nk} = \frac{1}{\sqrt{N}} e^{2\pi i k n/N}$$

$\Rightarrow$ ϕ_n er søylene i F_N^{-1}

siden F_N er unitær: $F_N^{-1} = F_N^H = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{pmatrix}^H$

$$= \begin{pmatrix} \overline{\phi_1} \\ \overline{\phi_2} \\ \vdots \\ \overline{\phi_n} \end{pmatrix} \rightarrow \frac{1}{\sqrt{N}} e^{-2\pi i k n/N}$$

Eks. 21 $\vec{x}: x_k = \cos(2\pi 5k/N)$ $\vec{y}: y_k = \sin(2\pi 7k/N)$

$$F_N(2\vec{x} + 3\vec{y}) = 2F_N\vec{x} + 3F_N\vec{y}$$

$$= 2F_N\left(\frac{1}{2}\left(e^{2\pi i5k/N} + e^{-2\pi i5k/N}\right)\right)$$

$$+ 3F_N\left(\frac{1}{2i}\left(e^{2\pi i7k/N} - e^{-2\pi i5k/N}\right)\right) \quad n=-5$$

$\sqrt{N}\,\phi_5$

$$= F_N(e^{2\pi i5k/N}) + F_N e^{-2\pi i5k/N}$$

$$+ \frac{3}{2i}F_N(e^{2\pi i7k/N}) - \frac{3}{2i}F_N(e^{-2\pi i7k/N}$$

$e^{-2\pi i5k/N}$

$= e^{2\pi i(N-5)k/N}$ $\underbrace{}_{2\pi i Nk/N = 2\pi i k}$

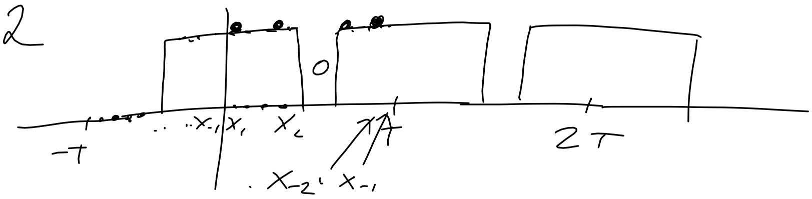
$$= F_N(\sqrt{N}\phi_5) + F_N(\sqrt{N}\phi_{N-5}) + \frac{3}{2i}F_N(\sqrt{N}\phi_7) - \frac{3}{2i}F_N(\sqrt{N}\phi_{N-7})$$

$$= \sqrt{N}\left(F_N\phi_5 + F_N\phi_{N-5} - \frac{3i}{2}F_N\phi_7 + \frac{3i}{2}F_N\phi_{N-7}\right)$$

$$= \sqrt{N}\left(\vec{e}_5 + \vec{e}_{N-5} - \frac{3i}{2}\vec{e}_7 + \frac{3i}{2}\vec{e}_{N-7}\right)$$

$$= \sqrt{N}\,(0,0,0,0,0,\underset{5}{1},0,\underset{7}{-\tfrac{3i}{2}},0,\ldots,\underset{N-7}{\tfrac{3i}{2}},0,\underset{N-5}{1},0,0,0,\underset{N-1}{0})$$

eks. 2.2



$$x_{-L} = \cdots x_{-1} = x_0 = x_1 = \cdots = x_L = 1 \quad , \text{ andre } x_k = 0$$

$$y_n = (\mathrm{DFT}\,\vec{x})_n = \sum_{k=0}^{L} 1 \cdot e^{-2\pi i k n/N} + \sum_{k=N-L}^{N-1} 1 \cdot e^{-2\pi i k n/N} \qquad k \to k-N$$

$$= \sum_{k=0}^{L} e^{-2\pi i k n/N} + \sum_{k=-L}^{-1} e^{-2\pi i k n/N} = \sum_{k=-L}^{L} e^{-2\pi i k n/N}$$

$$= a_0 \frac{1-k^{2L+1}}{1-k} = e^{2\pi i L n/N} \frac{1-(e^{-2\pi i n/N})^{2L+1}}{1-e^{-2\pi i n/N}}$$

$$= e^{2\pi i L n/N} \frac{e^{-\pi i n(2L+1)/N}\left(e^{\pi i n(2L+1)/N} - e^{-\pi i n(2L+1)/N}\right)\frac{1}{2i}}{e^{-\pi i n/N}\left(e^{\pi i n/N} - e^{-\pi i n/N}\right)\frac{1}{2i}}$$

$$= \frac{e^{2\pi i L n/N}\, e^{-\pi i n(2L+1)/N}}{e^{-\pi i n/N}} \frac{\sin(\pi n(2L+1)/N)}{\sin(\pi n/N)} = \underline{\underline{\frac{\sin(\pi n(2L+1)/N)}{\sin \pi n/N}}}$$

Vi ser at y_n ble reell.

Vi har også at $x_n = x_{N-n}$

Bevis for (1):

$$\left(\hat{x}\right)_{N-n} = \sum_{k=0}^{N-1} x_k e^{-2\pi i k(N-n)/N} = \sum_{k=0}^{N-1} y_k e^{2\pi i k n/N} \underbrace{e^{-2\pi i k}}_{1}$$

$$= \overline{\sum_{k=0}^{N-1} \cancel{x_k} e^{2\pi i k n/N}}$$

$$= \overline{\sum_{k=0}^{N-1} x_k e^{-2\pi i k n/N}}$$

$$= \overline{\left(\hat{x}\right)_n}$$

Bevis for (4)

$\vec{z}$: $z_k = x_{k-d}$

$$u = k-d \Rightarrow k = u+d$$

$$\left(\hat{z}\right)_n = \sum_{k=0}^{N-1} z_k e^{-2\pi i k n/N} = \sum_{k=0}^{N-1} x_{k-d} e^{-2\pi i k n/N} = \sum_{u=-d}^{N-1-d} x_u e^{-2\pi i (u+d)n/N}$$

$$= \sum_{k=0}^{N-1} x_k e^{-2\pi i (k+d)n/N} = e^{-2\pi i d n/N} \sum_{k=0}^{N-1} x_k e^{-2\pi i k n/N}$$

$$= e^{-2\pi i d n/N} \left(\hat{x}\right)_n$$

Eks. 2.5

Anta at $F_8(\vec{x}) = (1, \ldots, 8)$

Definer $\vec{z}$ ved at $z_k = e^{2\pi i 2k/8} x_k$

Hva blir $F_8(\vec{z})$?

$d = 2$

$$\left(F_8(\vec{z})\right)_n = \left(F_8(\vec{x})\right)_{n-2}$$

$$=$$

$$\Rightarrow \underline{F_8(\vec{z}) = (7, 8, 1, 2, 3, 4, 5, 6)}$$

Sammenheng mellom DFT-indeks n og frekvens ν

$$e^{2\pi i k n/N} \leftarrow e^{2\pi i n t/T} \qquad e^{2\pi i \nu t}$$

$$\frac{n}{T} = \nu$$

$$f_s = \frac{N}{T}$$

$$\Rightarrow T = \frac{N}{f_s}$$

$$\boxed{\frac{n f_s}{N} = \nu}$$