Definisjon 2.9

La R_N være antall regneoperasjoner (addisjoner/multipl.) for en algoritme (feks FFT), der N er dimensjonen på dataene (N i DFT_N)

Vi sier at algoritmen er av orden $f(N)$ (og skriver $O(f(N))$)

hvis antall operasjoner R_N vokser som $f(N)$, dvs

$$\lim_{N\to\infty} \frac{R_N}{f(N)} = 1$$

La M_N / A_N være antall komplekse multiplikasjoner / addisjoner i FFT av lengde N

$$y_0, \dots, y_{\frac{N}{2}-1} = DFT_{N/2}\, x^{(e)} + D_{N/2}\, DFT_{\frac{N}{2}}\, x^{(o)}$$

$$y_{\frac{N}{2}}, \dots, y_{N-1} = DFT_{\frac{N}{2}}\, x^{(e)} - D_{N/2}\, DFT_{N/2}\, x^{(o)}$$

diagonal, $\frac{N}{2} \times \frac{N}{2}$ $\Rightarrow$ $\frac{N}{2}$ mults.

$$M_N = 2 \cdot M_{\frac{N}{2}} + \frac{N}{2}, \quad A_N = 2A_{\frac{N}{2}} + N \quad \text{komplekse mult/adds}$$

$$(a+ib)(c+id) = ac - bd + i(ad+bc)$$

en kompleks mult = 4 reelle mult, og 2 reelle adds $\times 2$

totale antall reelle mults: $\boxed{M_N = 2M_{\frac{N}{2}} + 2N}$

reelle adds: $A_N = 2A_{\frac{N}{2}} + 2N + N = 2A_{\frac{N}{2}} + 3N$

Sett $x_r = M_{2^r}$ $\overset{N=2^r}{\Rightarrow}$ $x_r = 2x_{r-1} + 2\cdot 2^r$

$$x_r - 2x_{r-1} = 2\cdot 2^r$$

$$x_{r+1} - 2x_r = 2\cdot 2^{r+1} = 4\cdot 2^r \qquad x_2 = 0$$

vi trenger ikke gjøre noen multiplikasjoner for å regne ut DFT_4: elementene i DFT_4: $e^{-2\pi i kn/N} = e^{-\pi i kn/2} = i, -i, 1, -1$

$$\begin{bmatrix} \pm 1 & \pm i \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_{r+1} - 2x_r = 4\cdot 2^r \qquad x_2 = 0$$

x_r^h: $x_{r+1}^h - 2x_r^h = 0$ $\qquad r - 2 = 0 \Rightarrow r = 2$

$\Rightarrow \underline{x_r^h = C\cdot 2^r}$

x_r^p: Vi prøver $x_r^p = Ar2^r$

$$x_{r+1}^p - 2x_r^p = A(r+1)2^{r+1} - 2Ar2^r = 2^r(2A(r+1) - 2Ar)$$

$$= 2^r \cdot 2A = 4\cdot 2^r$$

$$\Rightarrow \underline{A = 2} \Rightarrow x_r^{(p)} = 2r2^r$$

generell løsning: $x_r = x_r^p + x_r^h = 2r2^r + C\cdot 2^r$

$x_2 = 0 \Rightarrow 0 = 2\cdot 2\cdot 2^2 + C\cdot 2^2 = 16 + 4C \Rightarrow C = -4$

$\Rightarrow x_r = 2r\cdot 2^r - 4\cdot 2^r$ $\qquad N = 2^r \Rightarrow \underline{r = \log_2 N}$

$$M_N = 2\log_2 N\cdot N - 4\cdot N = \underline{\underline{2N\log_2 N - 4N}}$$

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multiplikasjon med en kompleks $N\times N$ matrise A

$$(A\vec{x})_n = \underbrace{\sum_{i=1}^{N} a_{ni} x_i}_{N \text{ mults, } N-1 \text{ adds}}$$

$\Rightarrow N^2$ mults, komplekse $\Downarrow$ $\underline{4N^2}$ reelle mults.

$$z_n = \frac{1}{4}\left(x_{n-1} + 2x_n + x_{n+1}\right)$$

$$z_{n+N} = \frac{1}{4}\left(x_{n+N-1} + 2x_{n+N} + x_{n+N+1}\right)$$

$$= \frac{1}{4}\left(x_{n-1} + 2x_n + x_{n+1}\right) = z_n$$

$\Rightarrow \vec{z}$ er periodisk med samme periode N.

matrisen for filteret (kalles for S) $\quad (x_0, \dots, x_{N-1}) \to (z_0, \dots, z_{N-1})$

rad 0: $\quad z_0 = \frac{1}{4}\left(x_{-1} + 2x_0 + x_1\right) = \frac{1}{4}\left(\overset{x_{-1}}{x_{N-1}} + 2x_0 + x_1\right)$

$$= \frac{1}{4} \underbrace{\begin{bmatrix} \underset{0}{2} & \underset{1}{1} & 0 & \cdots & 0 & \underset{N-1}{1} \end{bmatrix}}_{\text{rad } 0 \text{ i } S} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$$z_1 = \frac{1}{4}\left(x_0 + 2x_1 + x_2\right) = \frac{1}{4} \underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 & \cdots & 0 \end{bmatrix}}_{\text{rad } 1 \text{ i } S} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \cdot \end{bmatrix}$$

$$z_{N-1} = \frac{1}{4}\left(x_{N-2} + 2x_{N-1} + x_N\right) = \frac{1}{4}\left(x_{N-2} + 2x_{N-1} + x_0\right)$$

$$= \frac{1}{4} \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}}_{\text{rad } N-1 \text{ i } S.} \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

Fra matrise til fillernotasjon:

$$S = \begin{array}{c} t_0 \to \\ t_1 \to \\ \to \\ t_{-1} \to \end{array} \begin{pmatrix} 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \end{pmatrix}$$

$t_0 = 2,\ t_1 = 3,\ t_{-1} = 1$

$$\Rightarrow \underline{\underline{S = \{1, \underline{2}, 3\}}}$$

$$\{1, 2, 3, \underline{4}, 5, 6\}$$

Konvolusjon $\vec{t} \to (\dots 0, 0, \underline{t_0, \dots, t_{M-1}}, 0 \dots)$

$\vec{x} \to (\dots 0, x_0, \dots, x_{N-1}, 0 \dots)$

$(t * x)_n \sum_k t_k x_{n-k}$

$0 \le k \le M-1$

$0 \le n-k \le N-1$

$\Downarrow +k$

$\boxed{0 \le k \le n} \le N-1+k \le N-1+M-1$

$n \le N+M-2$

$\Rightarrow (t*x)_n \neq 0$ kun for $n = 0, \dots, N+M-2$ $\quad n \geq 0$

konv av $t \in \mathbb{R}^M$ med $x \in \mathbb{R}^N \to t * x \in \mathbb{R}^{N+M-1}$

$(t_0 + t_1 x + \dots t_{M-1} x^{M-1})(x_0 + x_1 x + \dots + x_{N-1} x^{N-1})$

$= \sum_{n=0}^{N+M-2} \Big(\underbrace{\sum_{k=0}^{n} t_k x_{n-k}}_{\text{conv.}}\Big) x^n$

$(1+x)^m (1+x)^n = (1+x)^{n+m}$