$$(\phi_n)_k = \frac{1}{\sqrt{N}} e^{2\pi i kn/N}$$

$S = PDP^T$

$S = PDP^H$

ϕ_n egenvektorer for S

$$\Rightarrow S = \underbrace{[\phi_0, \phi_1, \ldots \phi_{N-1}]}_{\frac{1}{\sqrt{N}}\left[e^{2\pi i kn/N}\right]_{0\le k,n\le N-1} = F_N^H} \begin{bmatrix} \lambda_{S,0} & & 0 \\ & \ddots & \\ 0 & & \lambda_{S,N-1} \end{bmatrix} [\phi_0, \phi_1, \ldots, \phi_{N-1}]^H = \underline{F_N^H D F_N}$$

$\lambda_{S,n}$

Bevis for at $S_2 S_1 = S_1 S_2$:

$S_1 S_2 = F_N^H D_1 \underline{F_N F_N^H} D_2 F_N = F_N^H D_1 D_2 F_N$ siden $D_1 D_2 = D_2 D_1$

$S_2 S_1 = F_N^H D_2 \underline{F_N F_N^H} D_1 F_N = F_N^H D_2 D_1 F_N$

$S_1 S_2$ er også et filter:

$$S_1 S_2 \phi_n = S_1 \lambda_{S_2,n} \phi_n = \lambda_{S_2,n} S_1 \phi_n = \underbrace{\lambda_{S_2,n} \lambda_{S_1,n}}_{} \phi_n \Rightarrow S_1 S_2 \text{ er filter.}$$

egenverdiene til $S_1 S_2$ er $\lambda_{S_1,n} \lambda_{S_2,n}$

Bevis for teorem 3.13

Vi betegner tidsforsinkelse med d elementer med matrisen E_d

($\vec{x}$ komponenter x_k, så vil $E_d\vec{x} = \vec{z}$, der $z_k = x_{k-d}$)

$E_d\vec{e}_0 = \vec{e}_d$ $\qquad E_d\vec{e}_k = \vec{e}_{(k+d) \bmod N}$

$E_d = d \rightarrow \begin{pmatrix} 0 & & & \ddots \\ \vdots & \ddots & & \\ 1 & & \ddots & \\ & \ddots & & 0 \end{pmatrix}$ to diagonaler $\neq 0$

ser at E_d er sirkulant Toeplitz.

tidsinvarians $\Leftrightarrow$ $(S\vec{z} = \vec{w})$ $\Leftrightarrow$ $SE_d = E_dS$

$\underline{SE_d\vec{x}} = E_d\vec{y} = \underline{E_dS\vec{x}}$

E_d er et filter: $E_d(e^{2\pi ikn/N}) = e^{2\pi i(k-d)n/N} = e^{-2\pi idn/N}e^{2\pi ikn/N}$

$\Rightarrow E_d\phi_n = e^{-2\pi idn/N}\phi_n \Rightarrow E_d$ er et filter.

<u>Bevis 1. → 3.</u> Anta S filter. Siden E_d også er et filter så er $SE_d = E_dS \Leftrightarrow S$ tidsinvariant.

<u>Bevis 3. → 2.</u> S tidsinvariant: $\qquad E_d\vec{e}_0 = \vec{e}_d$

søyle d i S: $\vec{s}_d = S\vec{e}_d = SE_d\vec{e}_0 = E_dS\vec{e}_0 = E_d\vec{s}_0$

Dermed får vi søyle d i S ved å forsinke søyle 0 i S med d elementer

$\Rightarrow S$ er sirkulant Toeplitz.

<u>Bevis 2. → 1.</u>: Anta S sirkulant Toeplitz.

$$S = \sum_{d=0}^{N-1} s_d E_d$$

siden E_d er filter $\Rightarrow \sum_{d=0}^{N-1} s_d E_d$ også et filter.

(hvis flere matriser har samme egenvektorer, så vil summen av dem ha de samme egenvektorene). ∎

<u>Bevis for teorem 3,14:</u>

$$DFT_N \vec{s} = \sqrt{N} F_N \vec{s} = \sqrt{N} F_N S \vec{e}_0 = \sqrt{N} F_N F_N^H D F_N \vec{e}_0$$

$$= \sqrt{N} D \underbrace{F_N \vec{e}_0}_{\text{søyle } 0 \text{ i } F_N} = \frac{1}{\sqrt{N}} e^{-2\pi i \cdot 0 \cdot n/N} = \frac{1}{\sqrt{N}}$$

$$= \sqrt{N} \begin{bmatrix} \lambda_{S,0} & & 0 \\ & \ddots & \\ 0 & & \lambda_{S,N-1} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{N}} \\ \vdots \\ \frac{1}{\sqrt{N}} \end{bmatrix} = \begin{bmatrix} \lambda_{S,0} & & 0 \\ & \ddots & \\ 0 & & \lambda_{S,N-1} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_{S,0} \\ \vdots \\ \lambda_{S,N-1} \end{bmatrix} = \underline{\vec{\lambda}_S}$$

$$\Rightarrow \underline{\vec{s} = IDFT_N \vec{\lambda}_S}$$

Eks. 3.10

La oss se på filteret gitt ved $z_n = \frac{1}{4}(x_{n-1} + 2x_n + x_{n+1})$ (3.1)

$t_1 = t_{-1} = \frac{1}{4} \quad t_0 = \frac{1}{2}$

$\mathrm{DFT}_N \vec{s} = \sum_{k=0}^{N-1} s_k e^{-2\pi i k n/N}$

$s_0 = \frac{1}{2},\ s_1 = \frac{1}{4} \qquad s_{N-1} = \frac{1}{4}$

alle andre $s_n = 0$

$$\lambda_{S,n} = \mathrm{DFT}_N \vec{s} = \mathrm{DFT}_N \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4} \end{bmatrix} = \frac{1}{2} + \frac{1}{4} e^{-2\pi i n/N} + \frac{1}{4} e^{-2\pi i (N-1)n/N}$$

$$e^{-2\pi i N/N} = e^{-2\pi i n} = 1$$

$$= \frac{1}{2} + \underline{\frac{1}{4} e^{-2\pi i n/N} + \frac{1}{4} e^{2\pi i n/N}}$$

$$= \frac{1}{2} + \frac{1}{2}\cos(2\pi n/N) \qquad \geq 0$$

$$\leq 1$$

Eks 3.11

$S = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$, sirkulant Toeplitz.

$DFT_2 = \left(e^{-2\pi i kn/2}\right) = \left(e^{-\pi i kn}\right)$

egenverdiene kan da fåes med

$\lambda_S = DFT_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

egenvektor for $\lambda_0 = 5$: $\frac{1}{\sqrt{2}}(1,1)$: $\phi_0 = \frac{1}{\sqrt{2}} e^{2\pi i kn/2}$

$\lambda_1 = 3$: $\frac{1}{\sqrt{2}} e^{2\pi i kn/2} = \frac{1}{\sqrt{2}}(1,-1)$

$\lambda I - S = \begin{pmatrix} \lambda-4 & -1 \\ -1 & \lambda-4 \end{pmatrix}$ $\quad \det(\lambda I - S) = (\lambda-4)^2 - 1 = \lambda^2 - 8\lambda + 15$

$\Rightarrow \lambda = \frac{8 \pm \sqrt{64-60}}{2} = \frac{8 \pm 2}{2} = 4 \pm 1.$

$\lambda_0 = 5$:

$0 = (S - 5I)\vec{v}_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\vec{v}_0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x = y$

Eks. 3.12

$x_k = \cos(2\pi 5k/N)$, $0 \le k < N$

$= \frac{1}{2}\left(e^{2\pi i 5k/N} + e^{-2\pi i 5k/N}\right) = \frac{\sqrt{N}}{2}\left(\frac{1}{\sqrt{N}} e^{2\pi i 5k/N} + \frac{1}{\sqrt{N}} e^{2\pi i (\underline{N}-5)k/N}\right)$

$= \frac{\sqrt{N}}{2}\left(\phi_5 + \phi_{N-5}\right)$

$S\vec{x} = \frac{\sqrt{N}}{2}\left(S\phi_5 + S\phi_{N-5}\right) = \frac{\sqrt{N}}{2}\left(\underline{\lambda_{S,5}}\phi_5 + \underline{\lambda_{S,N-5}}\phi_{N-5}\right)$

$S: \quad z_n = \frac{1}{6}\left(x_{n+2} + 4x_{n+1} + 6x_n + 4x_{n-1} + x_{n-2}\right)$

$t_{-2} = \frac{1}{6} \quad t_{-1} = \frac{2}{3} \quad \underline{t_0 = 1} \quad \underline{t_1 = \frac{2}{3}} \quad \underline{t_2 = \frac{1}{6}}$

hvis $N = 8$ så blir

$$S = \frac{1}{6}\begin{pmatrix} 6 & 4 & 1 & \cdots \\ 4 & 6 & 4 & \cdots \\ 1 & 4 & 6 & \\ 0 & 1 & 4 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ 4 & 1 & 0 & \end{pmatrix}$$

$\Rightarrow s_0 = t_0 = 1$

$s_1 = t_1 = \frac{2}{3}$

$s_2 = t_2 = \frac{1}{6}$

$s_{N-2} = t_{-2} = \frac{1}{6}$

$s_{N-1} = t_{-1} = \frac{2}{3}$

$\lambda_{S,n} = \left(DFT_N \vec{s}\right)_n = s_0 + s_1 e^{-2\pi i n/N} + s_2 e^{-2\pi i 2n/N} + s_{N-2} e^{-2\pi i (\underline{N}-2)n/N} + s_{N-1} e^{-2\pi i (\underline{N}-1)n/N}$

$= 1 + \underline{\frac{2}{3} e^{-2\pi i n/N}} + \underline{\frac{1}{6} e^{-2\pi i 2n/N} + \frac{1}{6} e^{2\pi i 2n/N}} + \underline{\frac{2}{3} e^{2\pi i n/N}}$

$= 1 + \frac{4}{3}\cos(2\pi n/N) + \frac{1}{3}\cos(4\pi n/N)$

$\lambda_{S,5} = 1 + \frac{4}{3}\cos(10\pi/N) + \frac{1}{3}\cos(20\pi/N)$

$\lambda_{S,N-5} = 1 + \frac{4}{3}\cos(2\pi(\underline{N}-5)/N) + \frac{1}{3}\cos(4\pi(\underline{N}-5)/N)$

$= 1 + \frac{4}{3}\cos(-10\pi/N) + \frac{1}{3}\cos(-20\pi/N)$

$= 1 + \frac{4}{3}\cos(10\pi/N) + \frac{1}{3}\cos(20\pi/N) = \lambda_{S,5}$

$S\vec{x} = \frac{\sqrt{N}}{2}\left(\underline{\lambda_{S,5}}\phi_5 + \underline{\lambda_{S,N-5}}\phi_{N-5}\right)$

$= \lambda_{S,5}\left(\frac{\sqrt{N}}{2}\phi_5 + \frac{\sqrt{N}}{2}\phi_{N-5}\right) = \lambda_{S,5}\vec{x}$

$= \left(1 + \frac{4}{3}\cos(10\pi/N) + \frac{1}{3}(\cos(20\pi/N)\right)\vec{x}$

oppgave 3.13

er tidsreversering et filter?

tidsreversering av $\vec{x} \in \mathbb{R}^N$ $\quad (x_0, \dots, x_{N-1}) \to (x_{N-1}, x_{N-2}, \dots, x_0)$

$$\vec{e}_0 \to \vec{e}_{N-1}$$
$$\vec{e}_1 \to \vec{e}_{N-2}$$
$$\vdots$$
$$\vec{e}_{N-1} \to \vec{e}_0$$

matrisen blir:

$$\begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

på en hoveddiagonal her er det en ener, og resten nuller, så den er ikke Toeplitz, så da er den ikke et filter.

(se også på $(x_0, \dots, x_{N-1}) \to (x_0, 0, x_2, 0, \dots)$)