5.6 Generalisering fra stykkevis konstante/lineære wavelets:

Def 5.29 En multiresolusjonsanalyse (MRA) er en sekvens av underrom

$$V_0 \subset V_1 \subset \dots \subset V_m \subset \dots$$

slik at

1. "Alle" funksjoner kan tilnærmes vilkårlig fra V_m, bare m velges stor nok.
2. $f(t) \in V_0 \Leftrightarrow f(2^m t) \in V_m \longrightarrow (\phi \to \phi_{m,n})$
3. $f(t) \in V_0 \Leftrightarrow f(t-n) \in V_0$ alle n
4. Det finnes en funksjon ϕ s.a. $\Phi_0 = \{\phi(t-n)\}_{n=0}^{N-1}$ er en basis for V_0

Ingen krav om at Φ_0 er en ortonormal basis.

Utfordring: Definere hva detaljrommene W_m er, og hva ψ er

$$V_{m+1} = V_m \oplus W_m$$

Seksjon 6.1

DWT/IDWT uttrykt ved hjelp av filtre.

kjernetransformasjonen for stykkevis konstante:

$$P_{C_1 \leftarrow \Phi_1} \text{ (DWT)} \quad P_{\Phi_1 \leftarrow C_1} \text{ (IDWT)}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \\ & & 0 \end{pmatrix} \qquad \|$$

kjernetransformasjonen for stykkevis lineære:

$$\sqrt{2} \begin{pmatrix} 1 & 0 & 0 & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & -\frac{1}{2} & \\ 0 & \vdots & 0 & \\ -\frac{1}{2} & \cdot & \cdot & \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & & \\ \frac{1}{2} & 1 & \frac{1}{2} & \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \frac{1}{2} & \\ 0 & \cdot & 0 & \\ \frac{1}{2} & \cdot & & \end{pmatrix} \quad \text{(IDWT)}$$

Bevis for teorem 6.3

La H være matrisen $\overset{\text{DWT}}{}$

$$\begin{pmatrix} (H_0)_{0,:} \\ (H_1)_{1,:} \\ (H_0)_{2,:} \\ (H_1)_{3,:} \\ \vdots \end{pmatrix}$$

Siden H og H_0 har de samme partallsradene så er

$$(H_0 c_m)_k = (H c_m)_k \quad \text{for } k \text{ partall}$$

— " — H_1 —————————— " — oddetallsradene så er

$$(H_1 c_m)_k = (H c_m)_k \quad \text{for } k \text{ oddetall.}$$

Bevis teorem 6.5

kjernen i IDWT = $P_{\Phi_1 \leftarrow C_1}$

$$C_m = G\begin{pmatrix} c_{m-1,0} \\ w_{m-1,0} \\ c_{m-1,1} \\ w_{m-1,1} \\ \vdots \end{pmatrix} = G\left(\begin{pmatrix} c_{m-1,0} \\ 0 \\ c_{m-1,1} \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 \\ w_{m-1,0} \\ 0 \\ w_{m-1,1} \\ \vdots \end{pmatrix}\right) = G\begin{pmatrix} c_{m-1,0} \\ 0 \\ c_{m-1,1} \\ 0 \\ \vdots \end{pmatrix} + G\begin{pmatrix} 0 \\ w_{m-1,0} \\ 0 \\ w_{m-1,1} \\ \vdots \end{pmatrix}$$

$$= G_0\begin{pmatrix} c_{m-1,0} \\ 0 \\ c_{m-1,1} \\ 0 \\ \vdots \end{pmatrix} + G_1\begin{pmatrix} 0 \\ w_{m-1,0} \\ 0 \\ w_{m-1,1} \\ \vdots \end{pmatrix} =$$

Filtrene for den stykkevis konstante waveleten.

$$H = G = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & & & \\ 1 & -1 & 0 & & 0 \\ 0 & & 1 & 1 & 0 \\ & & 1 & -1 & \\ 0 & & 0 & & \cdots \end{pmatrix}$$

→ antar et partall antall rader/søyler

$$H_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 \cdots 0 \\ \vdots & & & & \\ 0 & & \cdots & 0 & 1 \quad 1 \end{pmatrix}, \quad H_0 = \frac{1}{\sqrt{2}}\{1, \underline{1}\}$$

$$H_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \vdots & & & & \\ 0 & \cdots & 0 & 0 & 1 \quad -1 \end{pmatrix}, \quad H_0 = \frac{1}{\sqrt{2}}\{\underline{-1}, 1\}$$

$$G_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & \\ 1 & 1 & \\ 0 & 1 & \\ \vdots & 0 & \cdots \\ 0 & \vdots & \\ & 0 & \end{pmatrix}, \quad G_0 = \frac{1}{\sqrt{2}}\{\underline{1}, 1\}$$

$$G_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 & \\ 0 & -1 & \\ \vdots & 0 & \cdots \\ 0 & \vdots & \end{pmatrix}, \quad G_1 = \frac{1}{\sqrt{2}}\{1, \underline{-1}\}$$

$$\lambda_{G_0}(\omega) = \frac{1}{\sqrt{2}}(1 + e^{-i\omega}) = e^{-i\frac{\omega}{2}}(e^{i\frac{\omega}{2}} + e^{-i\omega/2}) \qquad |\lambda_{G_0}(\omega)| = \sqrt{2}\cos\frac{\omega}{2}$$

$$\lambda_{G_1}(\omega) = \frac{1}{\sqrt{2}}(e^{i\omega} - 1) \qquad |\lambda_{G_1}(\omega)| = \sqrt{2}\sin\frac{\omega}{2}$$

$$\Rightarrow |\lambda_{G_0}(0)| = \sqrt{2}, \quad \lambda_{G_0}(\pi) = 0 \Rightarrow G_0 \text{ lowpass}$$

$$\lambda_{G_1}(0) = 0, \quad |\lambda_{G_1}(\pi)| = \sqrt{2} \Rightarrow G_1 \text{ høypass}$$

Filtrene for den stykkevis lineære waveleten

$$G = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & & \\ \frac{1}{2} & 1 & & \\ 0 & 0 & \cdots & \\ \vdots & \vdots & & \\ \frac{1}{2} & \vdots & \cdots & \end{pmatrix} \qquad G_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \\ \frac{1}{2} & \\ 0 & \cdots \\ \vdots & \\ \frac{1}{2} & \end{pmatrix}$$

$G_0 = \frac{1}{\sqrt{2}}\{\frac{1}{2}, \underline{1}, \frac{1}{2}\}$

$\lambda_{G_0}(\omega) = \frac{1}{\sqrt{2}}\left(\frac{1}{2}e^{i\omega} + 1 + \frac{1}{2}e^{-i\omega}\right)$

$= \frac{1}{\sqrt{2}}(1 + \cos\omega)$

$\lambda_{G_0}(0) = \sqrt{2}, \quad \lambda_{G_0}(\pi) = 0$

$\Rightarrow G_0$ lavpass

$$G_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ 0 & 0 & \cdots \\ \vdots & \vdots & \\ \vdots & 0 & \end{pmatrix} \Rightarrow G_1 = \frac{1}{\sqrt{2}}\{\underline{1}\}$$

$\lambda_{G_1}(\omega) = \frac{1}{\sqrt{2}}$

hverken lavpass eller høypass

$$H = \sqrt{2}\begin{pmatrix} 1 & 0 & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & -\frac{1}{2} & \\ 0 & \vdots & & \\ -\frac{1}{2} & 0 & & \end{pmatrix} \qquad H_0 = \sqrt{2}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & & & \end{pmatrix} \Rightarrow H_0 = \sqrt{2}\{\underline{1}\}$$

$\lambda_{H_0}(\omega) = \sqrt{2}$

hverken lavpass eller høypass

$$H_1 = \sqrt{2}\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \cdots & 0 \\ 0 & & & & \\ -\frac{1}{2} & \cdots & & -\frac{1}{2} & 1 \end{pmatrix} \Rightarrow H_1 = \sqrt{2}\{-\frac{1}{2}, \underline{1}, -\frac{1}{2}\}$$

$\lambda_{H_1}(\omega) = \sqrt{2}(1 - \cos\omega)$

$\lambda_{H_1}(0) = 0, \quad \lambda_{H_1}(\pi) = \underline{2\sqrt{2}}$

H_1 høypass

(se oppg 6.14 – 6.17)