$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$\approx \frac{f(x+h) - f(x-h)}{2h} \qquad (h = 1)$$

sek. 9.4.

Bevis for Teorem 9.16

Anta $\sum \alpha_{i,j} (\vec{v}_i \otimes \vec{w}_j) = 0$

Sett $\vec{h}_i = \sum_{j=0}^{N-1} \alpha_{i,j} \vec{w}_j \in \mathbb{R}^N$

$\sum_j \alpha_{ij} (\vec{v}_i \otimes \vec{w}_j) = \vec{v}_i \otimes \left(\sum_{j=0}^{N-1} \alpha_{ij} w_j \right) = \vec{v}_i \otimes \vec{h}_i$

$\Rightarrow \sum_{i,j} \alpha_{ij} (\vec{v}_i \otimes \vec{w}_j) = \sum_i \vec{v}_i \otimes \vec{h}_i = \underbrace{\sum_i \vec{v}_i \vec{h}_i^T}$

søyle k i denne er $\sum_i \vec{v}_i h_{i,k} = 0$

$\vec{v}_i$ lin uavh $\Rightarrow h_{i,k} = 0$ alle k

$\Downarrow$

$\vec{h}_i = 0$

$\Rightarrow \sum_j \alpha_{ij} \vec{w}_j = 0 \Rightarrow \alpha_{ij} = 0$ alle j

$\Rightarrow \alpha_{ij} = 0$ alle i og j.

$\Rightarrow$ $v_i \otimes w_j$ er lineært uavhengige, basis for alle matriser.

(mn basisvektorer her, og $L_{M,N}(\mathbb{R})$ er mn-dimensjonalt)

$\Rightarrow \{ v_i \otimes w_j \}$ basis.

Bevis for teorem 9.18: $\mathcal{B}_i = \{b_{ik}\}_k$, $\mathcal{C}_i = \{c_{ik}\}_k$

$\mathcal{B}_1 \otimes \mathcal{B}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$

$$b_{1i} \otimes b_{2j} = \left(\sum_k (S_1)_{ki} c_{1k}\right) \otimes \left(\sum_l (S_2)_{lj} c_{2l}\right)$$

$$= \sum_{k,l} (S_1)_{ki} (S_2)_{lj} c_{1k} \otimes c_{2l}$$

$$= \sum_{k,l} (S_1)_{ki} (S_2^T)_{jl} c_{1k} \otimes c_{2l} \longrightarrow (S_1 E_{ij} S_2^T)_{jl}, \quad E_{ij} = e_i e_j^T$$

$$= \sum_{kl} (S_1 e_i e_j^T S_2^T)_{kl} c_{1k} \otimes c_{2l}$$

$$= \sum_{k,l} \underbrace{(S_1 (e_i \otimes e_j) S_2^T)_{kl}}_{X} c_{1k} \otimes c_{2l}$$

$\Rightarrow$ koordinatmatrisen til $b_{1i} \otimes b_{2j}$ i $\mathcal{C}_1 \otimes \mathcal{C}_2$ er $S_1 (e_i \otimes e_j) S_2^T$

Hvis S er koordinatskiftet fra $\mathcal{B}_1 \otimes \mathcal{B}_2$ til $\mathcal{C}_1 \otimes \mathcal{C}_2$ så blir da

$$S(e_i \otimes e_j) = S_1 (e_i \otimes e_j) S_2^T$$

$\Rightarrow$ $\underline{S(X) = S_1 X S_2^T}$ også for alle X